



Department of Mathematics
Faculty of Sciences
Ghent University

Contributions to the linear and non-linear theory of ultradistributions

Andreas Debrouwere

Supervisors:

Prof. Dr. Hans Vernaeve

Prof. Dr. Jasson Vindas Díaz

Dissertation submitted in fulfillment of the
requirements for the degree of Doctor in Science:
Mathematics

Academic Year 2017 – 2018

Preface

This dissertation is devoted to the study of three topics in the theory of ultradifferentiable functions and ultradistributions. More precisely, we develop a non-linear theory of ultradistributions and infrahyperfuctions, study the topological properties of convolutor spaces, and introduce two new classes of weighted spaces of analytic functions and investigate their duals. Since these topics are thematically rather unrelated, they will be presented independently of each other. However, it is our belief that the way we approach them may be considered as the common ground in our work. Namely, we shall systematically employ the idea of regarding function spaces as locally convex spaces and often use abstract functional analytic tools in our arguments. This perspective turned out to be extremely useful on both a conceptual and a technical level.

In Part I, a non-linear theory of ultradistributions and infrahyperfuctions is developed. Most notably, we construct a differential algebra that contains the space of hyperfunctions as a linear differential subspace and in which the pointwise multiplication of real analytic functions is preserved, thereby fully settling a question posed by Oberguggenberger [123, p. 286, Problem 27.2]. This part is based on the papers [38, 39] (joint work H. Vernaev and J. Vindas), [40] (joint work with J. Vindas), and [37] (joint work with E. A. Nigsch).

We study topological properties of convolutor spaces of Gelfand-Shilov spaces in Part II. Both the smooth and the ultradifferentiable case are considered. In particular, we answer the question posed after [50, Thm. 3.3], which in fact was the original motivation for our investigations. This part is based on the papers [42, 43, 44] (joint work with J. Vindas).

Finally, in Part III, we introduce two new classes of spaces of analytic functions with very fast decay in strips of the complex plane with respect to a weight function. Their duals generalize the spaces of Fourier hyperfunctions and Fourier ultrahyperfunctions. An analytic representation theory for their duals is developed and applied to characterize the non-triviality of these function spaces in terms of the growth order of the defining weight function. This partially solves an important and long-standing open question concerning the non-triviality of Gelfand-Shilov spaces [63, Chap. 1]. This part is based on the paper [41] (joint work with J. Vindas).

Ghent, April 2018

Andreas Debrouwere

Acknowledgements

It is a great pleasure to thank my supervisors Professor Jasson Vindas and Professor Hans Vernaevae. Hans, thank you for your support and interest. Jasson, thank you for your constant encouragement and for showing me how to do research. I very much enjoyed our countless lively discussions on mathematics and other (un)related topics. Our collaboration over the past 4 years has deeply influenced my perspective on mathematics. Finally, I would like to thank my parents, my brother, and my friends for creating the ideal environment to work on this dissertation and, at the same time, making me not think about it when I needed a break.

Contents

Preface	i
Acknowledgements	iii
I A non-linear theory of ultradistributions and infrahyperfunctions	
1 Introduction	9
2 Preliminaries	13
2.1 Generalities	13
2.2 Linear spaces of generalized functions	16
2.2.1 Weight sequences	17
2.2.2 Spaces of ultradifferentiable functions and their duals	21
2.2.3 Ultradistributions	24
2.2.4 Infrahyperfunctions	25
2.3 Projective and inductive spectra of locally convex spaces	32
2.3.1 Regularity conditions for (LF) -spaces	32
2.3.2 The derived projective limit functor	35
3 Special Colombeau algebras	39
3.1 Introduction	39

3.2	Special Colombeau algebras based on a locally convex space	40
3.3	A general scheme of construction	45
3.4	Algebras of generalized functions of class (M_p) and $\{M_p\}$	51
3.5	Impossibility results	57
4	Optimal embeddings of spaces of ultradistributions into differential algebras	61
4.1	Introduction	61
4.2	Construction of the embedding	62
4.3	Regular generalized functions of class (M_p) and $\{M_p\}$	68
4.4	Wave front sets	70
5	Solution to the first Cousin problem for vector-valued quasianalytic functions	75
5.1	Introduction	75
5.2	Topological properties of spaces of quasianalytic functions of Roumieu type	77
5.3	Vector-valued quasianalytic functions	82
5.3.1	The ε -product of (PLS) -spaces	82
5.3.2	ε -product representations	85
5.3.3	Topological properties of spaces of vector-valued quasianalytic functions of Roumieu type	88
5.4	The Cousin problem	90
5.4.1	Cohomology groups with coefficients in a sheaf	90
5.4.2	The scalar-valued case	92
5.4.3	The vector-valued case	96
6	Optimal embeddings of spaces of infrahyperfuctions into differential algebras	99
6.1	Introduction	99

6.2	Analytic cut-off sequences	101
6.3	Sheaf properties	104
6.4	Construction of the embedding	106
7	Diffeomorphism invariant Colombeau algebras	115
7.1	Introduction	115
7.2	The basic space	117
7.3	The quotient construction	119
7.4	Sheaf properties	124
7.4.1	Locally convex sheaves	125
7.4.2	Localizing regularization operators	126
7.4.3	Sheaves of non-linear extensions	136
7.5	Diffeomorphism invariant differential algebras containing spaces of ultradistributions	148
II	Topological properties of convolutor spaces	
8	Introduction	157
9	Preliminaries	161
9.1	Weighted inductive limits of spaces of continuous functions	161
9.1.1	The (LB) -case	162
9.1.2	The vector-valued case	165
9.1.3	The (LF) -case	168
9.2	The Gelfand-Shilov spaces $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$	169
9.3	The short-time Fourier transform	170
9.3.1	The STFT on $\mathcal{D}'(\mathbb{R}^d)$	171
9.3.2	The STFT on $\mathcal{S}'_{(A_p)}(M_p)(\mathbb{R}^d)$ and $\mathcal{S}'_{\{A_p\}}(M_p)(\mathbb{R}^d)$. .	177
10	The smooth case	181
10.1	Introduction	181

10.2	Weighted inductive limits of spaces of smooth functions	184
10.2.1	Regularity properties	185
10.2.2	Characterization via the STFT	188
10.3	On a class of weighted L^1 convolutor spaces	191
10.3.1	Characterization via the STFT	192
10.3.2	A predual	194
10.3.3	Topological properties	198
10.4	Applications	206
10.4.1	Weighted $\mathcal{D}'_{L^1}(\mathbb{R}^d)$ spaces	206
10.4.2	Convolutor spaces of Gelfand-Shilov spaces	207
11	The ultradifferentiable case	215
11.1	Introduction	215
11.2	Weighted inductive limits of spaces of ultradifferentiable functions	216
11.2.1	Definition and basic properties	217
11.2.2	Regularity properties	221
11.2.3	Characterization via the STFT	225
11.2.4	A projective description	228
11.3	On two classes of weighted convolutor spaces	232
11.3.1	Characterization via the STFT	233
11.3.2	Preduals	237
11.3.3	Topological properties	242
11.4	Applications	246
11.4.1	Characterization of tempered ultradistribution spaces via convolution averages	246
11.4.2	Convolutor spaces of the Gelfand-Shilov spaces $\mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ and $\mathcal{S}^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d)$	247

III Hyperfunctions and ultrahyperfunctions of fast growth

12	Introduction	253
13	Analytic representation theory and the non-triviality of certain spaces of analytic functions	255
13.1	Introduction	255
13.2	Weight functions	258
13.3	Analytic functions in a strip	263
13.4	Weighted spaces of analytic functions	268
13.5	Boundary values	272
13.5.1	Two general results on boundary values . .	272
13.5.2	Boundary values of analytic functions in spaces of ultrahyperfunctions of fast growth	275
13.5.3	Boundary values of analytic functions in spaces of hyperfunctions of fast growth	279
13.6	Local theory	283
13.6.1	Real support of ultrahyperfunctions of fast growth	283
13.6.2	Support of hyperfunctions of fast growth . .	286
13.7	Spaces of hyperfunctions and ultrahyperfunctions defined via subadditive weight functions	288
13.7.1	Subadditive weight functions	288
13.7.2	Spaces of analytic functions with rapid decay in strips	289
13.7.3	Boundary values	291
14	Boundary values of analytic functions in ultradistribution spaces of exponential type	295
14.1	Introduction	295
14.2	Laplace transform	297
14.3	Boundary values	299

A	Open problems	309
B	Nederlandstalige samenvatting	315
	Bibliography	316

Part I

A non-linear theory of ultradistributions and infrahyperfunctions

Chapter 1

Introduction

Over the past 60 years, the theory of distributions, founded by Schwartz [151], has proved to be a very powerful tool in various branches of mathematical analysis such as partial differential equations (PDE's), Fourier analysis, and asymptotic analysis. As an example, we mention the Malgrange-Ehrenpreis theorem that states that any non-zero constant coefficient linear PDE admits a distributional fundamental solution. However, distribution theory is inherently linear in nature and one cannot define a reasonable multiplication on the whole space of distributions [70, 123]. Even worse, the Schwartz impossibility result [149] asserts that, for $\Omega \subseteq \mathbb{R}^d$ open, there does not exist an associative and commutative algebra $\mathcal{A}(\Omega) = (\mathcal{A}(\Omega), +, \circ)$ with unity satisfying the ensuing natural conditions:

- (i) The space $\mathcal{D}'(\Omega)$ of distributions is linearly embedded into $\mathcal{A}(\Omega)$ and the constant function 1 is the unity in $\mathcal{A}(\Omega)$.
- (ii) $\mathcal{A}(\Omega)$ is a differential algebra, i.e. there are linear operators $\partial_i : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$, $i = 1, \dots, d$, satisfying Leibniz's rule. Moreover, the mappings ∂_i extend the usual action of the partial derivatives on $\mathcal{D}'(\Omega)$.
- (iii) $\circ|_{C(\Omega) \times C(\Omega)}$ coincides with the pointwise product of functions.

Moreover, this result remains valid if one replaces $C(\Omega)$ by $C^n(\Omega)$, for any fixed $n \in \mathbb{N}$, in (iii). Nonetheless, in the 1980's Colombeau [28, 29] showed that it is possible to construct such an algebra if one replaces (iii) by the weaker requirement

(iii)' $\circ|_{C^\infty(\Omega) \times C^\infty(\Omega)}$ coincides with the pointwise product of functions.

The pioneer work of Colombeau was the starting point of the non-linear theory of generalized functions, which has been a very active field of research ever since. This theory provides a natural framework for non-linear PDE's and linear PDE's with strongly singular data or coefficients [123] and have found numerous applications in connection with singular differential geometry and general relativity [70].

On the other hand, also for several natural linear problems, the space of distributions is not the suitable setting, e.g. Lewy [107, 79] constructed a linear PDE with smooth coefficients that does not admit a distributional solution while Colombini and Spagnolo showed that there are Cauchy problems for weakly hyperbolic linear PDE's with smooth coefficients that are not well-posed in the space of distributions [33]. Such considerations motivated the search for and study of spaces of linear generalized functions that are strictly larger than the space of distributions; the most prominent examples being the spaces of (non-quasianalytic) ultradistributions [141, 89, 14, 21] and the space of hyperfunctions [145, 117, 86]. For instance, under suitable conditions the above Cauchy problems become well-posed in certain spaces of ultradistributions [32, 31, 62] while the space of hyperfunctions is the natural setting for the treatment of linear PDE's with real analytic coefficients [145, 117, 20, 88]. Interestingly, Lewy's equation also has no hyperfunctional solution [146] but is solvable in the space of Silva tempered ultrahyperfunctions [124]; spaces of ultrahyperfunctions will be studied in Part III of this work.

The natural question then appears whether it is possible to develop a non-linear theory of ultradistributions and hyperfunctions. The first problem that needs to be addressed is the construction of differential algebras that contain these spaces of generalized functions as a linear differential subspace and in which, at the same time, the pointwise product of sufficiently regular functions is preserved. The construction of such algebras will be the main subject of the first part of this work. Moreover, by establishing an analogue of Schwartz's impossibility result, we shall show that our constructions are optimal with respect to the preservation of the pointwise multiplication of ordinary functions. For ultradistributions this is a well-studied problem, see e.g. [47, 135, 46, 8, 9, 67]. However, in all of the aforementioned works the embeddings are not optimal and, moreover, there is a rather unnatural distinction between ultradistributions of Beurling and Roumieu type. We shall resolve both of these issues in Chapters 4 and 7 via two different approaches. In the case of hyperfunctions, this problem was explicitly raised by Oberguggenberger [123, p. 286, Problem 27.2] who asked for the construction of a differential algebra containing the space of hyperfunctions as a linear differential subspace and the space of real analytic functions as a subalgebra. We shall construct such an algebra in Chapter 6. In fact, we shall solve this problem for general spaces of infrahyperfunctions, as introduced by Hörmander in his seminal work¹ [78]. A key result in our approach is the solution to the first Cousin problem for vector-valued quasianalytic functions, which we shall treat separately in Chapter 5. Finally, we point out that embeddings of the space of hyperfunctions in the unit circle into differential algebras were considered in [36, 157, 46, 47] but that, to the best of our knowledge, no prior work has been done in the general local case.

¹Hörmander uses the name quasianalytic distributions in [78]; the terminology infrahyperfunctions comes from [129, 48].

Chapter 2

Preliminaries

In this chapter, we fix the notation and introduce the spaces of linear generalized functions that will be employed throughout Part I. Furthermore, we discuss an important result about the derived projective limit functor for spectra of Montel (DF)-spaces that will play a major role in Chapter 5.

2.1 Generalities

Notations

We include 0 in the set of natural numbers \mathbb{N} . A multi-index is an element $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$. We write $|\alpha| = |\alpha_1| + \dots + |\alpha_d|$ for its length. The notation $|x|$ is also employed for the Euclidean norm of a vector $x \in \mathbb{R}^d$ but the distinction should always be clear from the context. We use the standard multi-index notation, namely, $\alpha! = \alpha_1! \cdots \alpha_d!$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, where $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, and $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}$, where $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$. The i -th partial derivative is denoted by ∂_i and we write $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $D^\alpha = (-i)^\alpha \partial^\alpha$. Next, let X be a topological space and $A \subseteq X$. We denote by \bar{A} the closure of A in X and by $\text{int } A$ the interior of A in X . Let $U \subseteq X$ be open. The notation $K \Subset U$ means that K is a

compact subset of U . In this regard, we point out that in Chapter 7 we shall write $V \subset\subset U$ to indicate that V is a relatively compact subset of U ; the latter notation shall be employed for open sets V and U only. The space of complex-valued continuous functions on X is denoted by $C(X)$. Finally, by an algebra \mathcal{A} we always mean an associative and commutative algebra over \mathbb{C} with unity. A derivation on \mathcal{A} is a linear operator on \mathcal{A} satisfying Leibniz's rule. A locally convex algebra is an algebra endowed with a locally convex topology for which the multiplication is jointly continuous.

Function and distribution spaces

The classical Lebesgue spaces on \mathbb{R}^d are denoted by $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. We fix the constants in the Fourier transform as follows

$$\mathcal{F}(f)(\xi) := \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx, \quad f \in L^1(\mathbb{R}^d).$$

Let $\Omega \subseteq \mathbb{R}^d$ be open and let $n \in \mathbb{N} \cup \{\infty\}$. The space of n -times continuously differentiable functions on Ω is denoted by $C^n(\Omega)$. For $n = \infty$ we shall sometimes use the alternative notation $\mathcal{E}(\Omega) = C^\infty(\Omega)$. Let K be a regular compact subset of \mathbb{R}^d , that is, $K = \overline{\text{int } K}$. We write $C^n(K)$ for the space consisting of all $\varphi \in C^n(\text{int } K)$ such that $\partial^\alpha \varphi$ can be continuously extended to the whole of K for all $|\alpha| < n + 1$. For finite n we set

$$\|\varphi\|_{C^n(K)} := \max_{|\alpha| \leq n} \max_{x \in K} |\partial^\alpha \varphi(x)|, \quad \varphi \in C^n(K).$$

If $n = 0$, we simply write $\|\cdot\|_{C^0(K)} = \|\cdot\|_{C(K)}$. We assume the reader is familiar with classical distribution theory [151, 156, 82]. As customary, we denote by $\mathcal{D}(\Omega)$ and $\mathcal{S}(\mathbb{R}^d)$ the space of compactly supported smooth functions in Ω and the space of rapidly decreasing smooth functions in \mathbb{R}^d , respectively, each endowed with their standard locally convex topology. Their dual spaces $\mathcal{D}'(\Omega)$ and $\mathcal{S}'(\mathbb{R}^d)$ are the space of distributions in Ω and the space of tempered

distributions in \mathbb{R}^d , respectively. The dual space $\mathcal{E}'(\Omega)$ may be identified with the subspace of $\mathcal{D}'(\Omega)$ consisting of all distributions with compact support in Ω .

Locally convex spaces

We shall frequently use techniques from the theory of locally convex spaces. Our main references are [97, 82, 156]. In the sequel, every locally convex space X (from now on abbreviated as l.c.s.) is assumed to be Hausdorff. We write X' for its topological dual and, unless explicitly stated otherwise, we endow this space with the strong topology. The set of continuous seminorms on X is denoted by $\text{csn}(X)$. Let Y be another locally convex space. We denote by $L(X, Y)$ the space of continuous linear mappings from X into Y and write $L_b(X, Y)$ ($L_\sigma(X, Y)$, respectively) to indicate that we endow this space with the topology of uniform convergence on the bounded sets of X (the topology of pointwise convergence on X , respectively). We presume the reader is familiar with the theory of (FS) - and (DFS) -spaces [89, 117], projective and inductive limits [97, 82, 89, 10, 164], duality theory [97, 82, 156], and topological tensor products [156, 92, 73].

Sheaf theory

The language of sheaves will be repeatedly used in Part I of this work. Although we shall only use some basic facts about sheaves, they will play an important conceptual role. Our main references are [22, 65]. For further reference, we recall the definition of a sheaf. Let X be a topological space. A presheaf \mathcal{F} (of vector spaces) on X assigns to each open set $U \subseteq X$ a vector space $\mathcal{F}(U)$ and gives, for every inclusion of open sets $V \subseteq U$, a linear mapping $\rho_{V,U} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ such that the identities $\rho_{W,U} = \rho_{W,V} \circ \rho_{V,U}$ and $\rho_{U,U} = \text{id}$ hold for all $W \subseteq V \subseteq U$. The elements of $\mathcal{F}(U)$ are called the sections of \mathcal{F} over U and $\rho_{V,U}$ are called the restriction

mappings. As customary, we shall sometimes write $|_V$ instead of $\rho_{V,U}$ if U and \mathcal{F} are clear from the context. A presheaf \mathcal{F} on X is said to be a sheaf on X if for all open subsets $U \subseteq X$ and all open coverings $(U_i)_i$ of U the following conditions are satisfied:

- (S1) If $\varphi \in \mathcal{F}(U)$ satisfies $\rho_{U_i,U}(\varphi) = 0$ for all i , then $\varphi = 0$.
- (S2) Let $\varphi_i \in \mathcal{F}(U_i)$ be such that $\rho_{U_i \cap U_j, U_i}(\varphi_i) = \rho_{U_i \cap U_j, U_j}(\varphi_j)$ for all i, j . Then, there exists $\varphi \in \mathcal{F}(U)$ such that $\rho_{U_i,U}(\varphi) = \varphi_i$ for all i .

Let $U \subseteq X$ be open. For $A \subseteq U$ we write $\Gamma_A(U, \mathcal{F})$ for the space consisting of all $\varphi \in \mathcal{F}(U)$ with $\text{supp } \varphi \subseteq A$. Furthermore, we set

$$\Gamma_c(U, \mathcal{F}) = \bigcup_{K \Subset U} \Gamma_K(U, \mathcal{F}).$$

We presume the reader is familiar with flabby, fine, and soft sheaves [22, 65].

2.2 Linear spaces of generalized functions

In this section, we present a brief summary of the basic definitions and properties of the main spaces of ultradifferentiable functions and generalized functions, namely, ultradistributions and (in-)hyperfunctions, to be considered in the sequel. We shall always work with the notion of ultradifferentiability defined via weight sequences¹ [89, 108]. In the case of ultradistributions, we closely follow Komatsu's fundamental paper [89] while, for in(hy)perfunctions, our main reference is Hörmander's paper [78], which is based on Martineau's duality method for hyperfunctions [109, 147]. For a clear exposition of Sato's original approach to hyperfunctions via cohomology groups [145] we refer to the book [117].

¹There are several other approaches to ultradifferentiability, e.g. due to Beurling and Björck [14] and Braun, Meise and Taylor [21].

2.2.1 Weight sequences

We start by collecting several useful facts concerning weight sequences. Let $(M_p)_{p \in \mathbb{N}}$ be a sequence of positive real numbers and define $m_p := M_p/M_{p-1}$, $p \geq 1$. We call M_p a *weight sequence* if $m_p \rightarrow \infty$. We will make use of some of the following conditions [89, 108, 18]:

$$(M.1) \quad M_p^2 \leq M_{p-1}M_{p+1}, \quad p \geq 1.$$

$$(M.2)' \quad M_{p+1} \leq C_0 H^{p+1} M_p, \quad p \in \mathbb{N}, \text{ for some } C_0, H \geq 1.$$

$$(M.2) \quad M_{p+q} \leq C_0 H^{p+q} M_p M_q, \quad p, q \in \mathbb{N}, \text{ for some } C_0, H \geq 1.$$

$$(M.2)^* \quad 2m_p \leq m_{Qp}, \quad p \geq 1, \text{ for some } Q \geq 1.$$

$$(M.3)' \quad \sum_{p=1}^{\infty} 1/m_p < \infty.$$

$$(M.3) \quad \sum_{p=q}^{\infty} 1/m_p < Cq/m_q, \quad q \geq 1, \text{ for some } C > 0.$$

$$(QA) \quad \sum_{p=1}^{\infty} 1/m_p = \infty.$$

$$(NE) \quad p! \leq Ch^p M_p, \quad p \in \mathbb{N}, \text{ for some } C, h > 0.$$

$$(NA) \quad p! \leq Ch^p M_p, \quad p \in \mathbb{N}, \text{ for all } h > 0 \text{ and a suitable } C = C_h > 0.$$

A weight sequence satisfying $(M.3)'$ is said to be *non-quasianalytic*; otherwise it is called *quasianalytic*. It is worth mentioning that $(M.1)$ and $(M.3)'$ imply (NA) [89, Lemma 4.1] while $(M.1)$ and $(M.3)$ imply $(M.2)^*$ [130, Prop. 1.1]. As customary, the relation $M_p \subset N_p$ between two weight sequences means that there are $C, h > 0$ such that $M_p \leq Ch^p N_p$, $p \in \mathbb{N}$. The stronger relation $M_p \prec N_p$ means that the latter inequality remains valid for every $h > 0$ and a suitable $C = C_h > 0$. For example, the conditions (NE) and (NA) can be rewritten as $p! \subset M_p$ and $p! \prec M_p$, respectively.

Remark 2.2.1. The symbols C_0 and H will *always* refer to the constants appearing in $(M.2)$ or $(M.2)'$ (depending on which of these two conditions is assumed).

The *associated function* of M_p is defined as

$$M(t) := \sup_{p \in \mathbb{N}} \log \frac{t^p M_0}{M_p}, \quad t > 0,$$

and $M(0) := 0$. The function M is increasing, $\log(1+t) = o(M(t))$, and $M \circ \exp$ is convex [89, p. 49]. The weight sequence M_p satisfies (M.1) if and only if

$$M_p = \sup_{t \geq 0} \frac{t^p M_0}{e^{M(t)}}$$

for all $p \in \mathbb{N}$ [89, Prop. 3.2]. Suppose that M_p satisfies (M.1). The above inversion formula allows one to characterize conditions on M_p in terms of its associated function M . In particular, we shall frequently use the following facts:

- ([89, Prop. 3.4]) If M_p satisfies $(M.2)'$, then

$$M(H^\lambda t) - M(t) \geq \lambda \log(t/C_0), \quad t \geq 0,$$

for all $\lambda > 0$.

- ([89, Prop. 3.6]) M_p satisfies $(M.2)$ if and only if

$$2M(t) \leq M(Ht) + \log C_0, \quad t \geq 0.$$

- ([18, Prop. 13]) Assume that M_p satisfies $(M.2)$. Then, M_p satisfies $(M.2)^*$ if and only if

$$M(2t) \leq H' M(t) + \log C'_0, \quad t \geq 0,$$

for some $C'_0, H' \geq 1$.

- ([89, Lemma 4.1]) M_p satisfies $(M.3)'$ if and only if

$$\int_0^\infty \frac{M(t)}{1+t^2} dt < \infty.$$

- ([89, Lemma. 3.8]) Let N_p be another weight sequence satisfying (M.1) and denote by N its associated function. Then, $M_p \subset N_p$ ($M_p \prec N_p$, respectively) if and only if

$$N(t) \leq M(ht) + \log C, \quad t \geq 0,$$

for some $C, h > 0$ (for all $h > 0$ and a suitable $C = C_h > 0$, respectively). In particular, M_p satisfies (NE) ((NA), respectively) if and only if $M(t) = O(t)$ ($M(t) = o(t)$, respectively).

In fact, for the direct implications in the above results it is not necessary to assume (M.1). For a more detailed account on the relation between weight sequences and their associated function we refer to [89, 108, 18].

Finally, we introduce Komatsu's family \mathfrak{R} [92]. We write \mathfrak{R} for the set consisting of all increasing positive real sequences $(h_j)_{j \in \mathbb{N}}$ with $h_j \rightarrow \infty$. This set is partially ordered and directed by the relation $h_j \preceq k_j$, which means that there is $j_0 \in \mathbb{N}$ such that $h_j \leq k_j$ for all $j \geq j_0$. Let M_p be a weight sequence with associated function M and let $h_j \in \mathfrak{R}$. We denote by M_{h_j} the associated function of the weight sequence $M_p \prod_{j=0}^p h_j$. The following three technical lemmas will be often used.

Lemma 2.2.2. ([92, Lemma 3.4]) *Let $(a_n)_n$ be a sequence of positive reals.*

- (i) *$\sup_n a_n / h^n < \infty$ for some $h > 0$ if and only if*

$$\sup_{n \in \mathbb{N}} \frac{a_n}{\prod_{j=0}^n h_j} < \infty$$

for all $h_j \in \mathfrak{R}$.

- (ii) *$\sup_n a_n h^n < \infty$ for all $h > 0$ if and only if*

$$\sup_{n \in \mathbb{N}} a_n \prod_{j=0}^n h_j < \infty$$

for some $h_j \in \mathfrak{R}$.

Lemma 2.2.3. ([138, Lemma 2.3]) *For every $h_j \in \mathfrak{R}$ there is $h'_j \in \mathfrak{R}$ such that $h'_j \preceq h_j$ and*

$$\prod_{j=0}^{p+q} h'_j \leq 2^{p+q} \prod_{j=0}^p h'_j \prod_{j=0}^q h'_j, \quad p, q \in \mathbb{N}.$$

Consequently, if M_p is a weight sequence satisfying $(M.2)'$ ($(M.2)$, respectively), then the weight sequence $M_p \prod_{j=0}^p h'_j$ satisfies $(M.2)'$ ($(M.2)$, respectively) with the constant $2H$ instead of H .

Lemma 2.2.4. *Let M_p be a weight sequence satisfying $(M.1)$ and $(M.2)'$ and let $f : [0, \infty) \rightarrow [0, \infty)$. Then, $f(t) = O(e^{M(t/h)})$ for all $h > 0$ if and only if $f(t) = O(e^{M_{h_j}(t)})$ for some $h_j \in \mathfrak{R}$.*

Proof. We only need to show the direct implication, the “if” part is clear. We first show that there is a subordinate function $\varepsilon : [0, \infty) \rightarrow [0, \infty)$ (which means that ε is continuous, increasing, and satisfies $\varepsilon(0) = 0$ and $\varepsilon(t) = o(t)$) such that $f(t) = O(e^{M(\varepsilon(t))})$. Condition $(M.2)'$ ensures that for all $h > 0$ there is $r > 0$ such that $f(t) \leq e^{M(t/h)}$ for all $t \geq r$. Hence, we can inductively define a sequence $(t_n)_{n \geq 1}$ with $t_1 = 0$ such that

$$f(t) \leq e^{M(t/(n+1))}, \quad t \geq t_n; \quad \frac{t_n}{n} \geq \frac{t_{n-1}}{n-1} + 1, \quad n \geq 2.$$

Denote by l_n the line through $(t_n, t_n/n)$ and $(t_{n+1}, t_{n+1}/(n+1))$. We define $\varepsilon(t) = l_n(t)$ for $t \in [t_n, t_{n+1})$. The function ε is subordinate and $f(t) \leq e^{M(\varepsilon(t))}$ for all $t \geq t_2$. Therefore, it suffices to show that, given an arbitrary subordinate function ε , there is a sequence $h_j \in \mathfrak{R}$ and $C > 0$ such that $M(\varepsilon(t)) \leq M_{h_j}(t) + C$ for all $t \geq 0$. By [89, Lemma 3.12] there is a weight sequence N_p satisfying $(M.1)$ such that $M_p \prec N_p$ and $M(\varepsilon(t)) \leq N(t)$ for all $t \geq 0$. Lemma 2.2.2(ii) implies that there is $h_j \in \mathfrak{R}$ such that $M_p \prod_{j=0}^p h_j \prec N_p$, whence the result follows. \square

Corollary 2.2.5. *Let $(a_n)_n$ be a sequence of positive reals. Then, $\sup_n a_n e^{-M(n/h)} < \infty$ for all $h > 0$ if and only if $\sup_n a_n e^{-M_{h_j}(n)} < \infty$ for some $h_j \in \mathfrak{R}$.*

2.2.2 Spaces of ultradifferentiable functions and their duals

Let M_p be a weight sequence. We write $M_\alpha = M_{|\alpha|}$, $\alpha \in \mathbb{N}^d$, and define M on \mathbb{R}^d as the radial function $M(x) = M(|x|)$, $x \in \mathbb{R}^d$. Let K be a regular compact subset of \mathbb{R}^d . For $h > 0$ we define $\mathcal{E}^{M_p, h}(K)$ as the Banach space consisting of all $\varphi \in C^\infty(K)$ such that

$$\|\varphi\|_{\mathcal{E}^{M_p, h}(K)} := \sup_{\alpha \in \mathbb{N}^d} \max_{x \in K} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|} M_\alpha} < \infty.$$

We set

$$\mathcal{E}^{(M_p)}(K) := \varprojlim_{h \rightarrow 0^+} \mathcal{E}^{M_p, h}(K), \quad \mathcal{E}^{\{M_p\}}(K) := \varinjlim_{h \rightarrow \infty} \mathcal{E}^{M_p, h}(K).$$

Let $\Omega \subseteq \mathbb{R}^d$ be open and let $(K_N)_{N \in \mathbb{N}}$ be an exhaustion by regular compact sets of Ω . We define

$$\mathcal{E}^{(M_p)}(\Omega) := \varprojlim_{N \in \mathbb{N}} \mathcal{E}^{(M_p)}(K_N), \quad \mathcal{E}^{\{M_p\}}(\Omega) := \varprojlim_{N \in \mathbb{N}} \mathcal{E}^{\{M_p\}}(K_N).$$

These definitions are independent of the chosen exhaustion by regular compact sets of Ω . The elements of $\mathcal{E}^{(M_p)}(\Omega)$ are called *ultra-differentiable functions of class (M_p) (of Beurling type) in Ω* while the elements of $\mathcal{E}^{\{M_p\}}(\Omega)$ are called *ultradifferentiable functions of class $\{M_p\}$ (of Roumieu type) in Ω* . In the sequel, we shall write $*$ instead of (M_p) or $\{M_p\}$ if we want to treat the Beurling and Roumieu case simultaneously. In addition, we shall frequently first state assertions for the Beurling case followed in parenthesis by the corresponding statements for the Roumieu case.

Remark 2.2.6. By Pringsheim's theorem the space $\mathcal{E}^{\{p!\}}(\Omega)$ consists precisely of all real analytic functions in Ω . In this case, we shall use the standard notation $\mathcal{A}(\Omega) = \mathcal{E}^{\{p!\}}(\Omega)$.

We shall often employ the following simple but useful Paley-Wiener type result.

Lemma 2.2.7. ([89, Lemma 3.3 part 2]) *Let M_p be a weight sequence satisfying (M.1) and let $h > 0$. Then, for every $\varphi \in L^1(\mathbb{R}^d)$ with $\widehat{\varphi}e^{M(\cdot/h)} \in L^1(\mathbb{R}^d)$ it holds that $\varphi \in C^\infty(\mathbb{R}^d)$ and that*

$$\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|} M_\alpha} \leq \frac{1}{2\pi M_0} \|\widehat{\varphi}e^{M(\cdot/h)}\|_{L^1}.$$

Next, we briefly discuss some of the algebraic and analytic properties of the spaces $\mathcal{E}^*(\Omega)$ and their relationship to the conditions on M_p introduced in Subsection 2.2.1:

- ([89, Lemma 2.7 and Thm. 2.8]) Assume that M_p satisfies (M.1). Then, $\mathcal{E}^*(\Omega)$ is a locally convex algebra under the pointwise multiplication. Moreover, for $h, k > 0$ and $K \Subset \mathbb{R}^d$ regular, we have that

$$\|\varphi\psi\|_{\mathcal{E}^{M_p, h+k}(K)} \leq M_0 \|\varphi\|_{\mathcal{E}^{M_p, h}(K)} \|\psi\|_{\mathcal{E}^{M_p, k}(K)}$$

for all $\varphi \in \mathcal{E}^{M_p, h}(K)$ and $\psi \in \mathcal{E}^{M_p, k}(K)$.

- ([89, Thm. 2.10]) Assume that M_p satisfies (M.2)'. Then, $\mathcal{E}^*(\Omega)$ is closed under the action of partial derivatives. Moreover, the mapping $\partial_i : \mathcal{E}^*(\Omega) \rightarrow \mathcal{E}^*(\Omega)$ is continuous for $i = 1, \dots, d$.

Condition (M.2) ensures that $\mathcal{E}^*(\Omega)$ is closed under the action of certain infinite order differential operators. We need some preparation. An entire function $P(z) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha z^\alpha$, $c_\alpha \in \mathbb{C}$, is said to be an *ultrapolynomial of class (M_p) (of class $\{M_p\}$)* if

$$\sup_{\alpha \in \mathbb{N}^d} \frac{c_\alpha M_\alpha}{h^{|\alpha|}} < \infty$$

for some $h > 0$ (for all $h > 0$). The associated infinite order differential operator $P(D) = \sum_{\alpha} c_\alpha D^\alpha$ is called an *ultradifferential operator of class (M_p) (of class $\{M_p\}$)*. We then have:

- ([89, Thm. 2.12]) Assume that M_p satisfies (M.2) and let $P(D) = \sum_{\alpha} c_{\alpha} D^{\alpha}$ be an ultradifferential operator of class $*$. Then, for each $\varphi \in \mathcal{E}^*(\Omega)$, the family $\{c_{\alpha} D^{\alpha} \varphi \mid \alpha \in \mathbb{N}^d\}$ is absolutely summable in $\mathcal{E}^*(\Omega)$. Moreover, the mapping $P(D) : \mathcal{E}^*(\Omega) \rightarrow \mathcal{E}^*(\Omega)$ given by $P(D)\varphi = \sum_{\alpha} c_{\alpha} D^{\alpha} \varphi$ is continuous.
- (Denjoy-Carleman theorem, [89, Thm. 4.2]) Assume that M_p satisfies (M.1). Then, M_p satisfies (QA) if and only if $\mathcal{E}^*(\Omega)$ does not contain any compactly supported function that is not identically zero, or equivalently, that a function $\varphi \in \mathcal{E}^*(\Omega)$ that vanishes on an open subset of Ω that meets every connected component of Ω , is necessarily identically zero.
- Assume that M_p satisfies (NE). Then, $\mathcal{A}(\Omega) \subseteq \mathcal{E}^{\{M_p\}}(\Omega)$ with continuous inclusion.
- Assume that M_p satisfies (NA). Then, $\mathcal{A}(\Omega) \subseteq \mathcal{E}^{(M_p)}(\Omega)$ with continuous inclusion.

Remark 2.2.8. Although we shall not use this fact, we remark that, under (M.1), (M.2), and (NE), a weight sequence M_p satisfies (M.2)* if and only if $\omega = M$ is a weight function in the sense of Braun, Meise, and Taylor [21] (as defined on [18, p. 426]). In such a case, we have that $\mathcal{E}^{(M_p)}(\Omega) = \mathcal{E}_{(\omega)}(\Omega)$ and $\mathcal{E}^{\{M_p\}}(\Omega) = \mathcal{E}_{\{\omega\}}(\Omega)$ [18, Thm. 14].

Finally, we consider the dual spaces $\mathcal{E}'^*(\Omega)$. In view of the above continuity properties, we can define the action of multiplication and (ultra)differential operators on $\mathcal{E}'^*(\Omega)$ via duality. More precisely:

- Assume that M_p satisfies (M.1). Let $\psi \in \mathcal{E}^*(\Omega)$ and let $f \in \mathcal{E}'^*(\Omega)$. We define $\psi f \in \mathcal{E}'^*(\Omega)$ via $\langle \psi f, \varphi \rangle := \langle f, \psi \varphi \rangle$ for all $\varphi \in \mathcal{E}^*(\Omega)$.
- Assume that M_p satisfies (M.2)'. Let $f \in \mathcal{E}'^*(\Omega)$. We define $\partial_i f \in \mathcal{E}'^*(\Omega)$ via $\langle \partial_i f, \varphi \rangle := -\langle f, \partial_i \varphi \rangle$ for all $\varphi \in \mathcal{E}^*(\Omega)$.

- Assume that M_p satisfies (M.2). Let $P(D)$ be an ultradifferential operator of class $*$ and let $f \in \mathcal{E}'^*(\Omega)$. We define $P(D)f \in \mathcal{E}'^*(\Omega)$ via $\langle P(D)f, \varphi \rangle := \langle f, P(-D)\varphi \rangle$ for all $\varphi \in \mathcal{E}^*(\Omega)$.

Our standard assumption $m_p \rightarrow \infty$ implies that $e^{-ix\xi} \in \mathcal{E}^*(\mathbb{R}_x^d)$ for $\xi \in \mathbb{C}^d$ fixed. Hence, we can define the Fourier transform of $f \in \mathcal{E}'^*(\mathbb{R}^d)$ as

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \langle f(x), e^{-ix\xi} \rangle, \quad \xi \in \mathbb{C}^d.$$

Then, \widehat{f} is an entire function that satisfies

$$\sup_{\xi \in \mathbb{C}^d} |\widehat{f}(\xi)| e^{-M(\xi/h)} < \infty$$

for some $h > 0$ (for all $h > 0$).

2.2.3 Ultradistributions

Let M_p be a weight sequence satisfying (M.1) and (M.3)'. For $h > 0$ and $K \Subset \mathbb{R}^d$ we write $\mathcal{D}_K^{M_p, h}$ for the Banach space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ with $\text{supp } \varphi \subseteq K$ such that $\|\varphi\|_{\mathcal{E}^{M_p, h}(K)} < \infty$. Let $\Omega \subseteq \mathbb{R}^d$ be open. We define

$$\mathcal{D}^{(M_p)}(\Omega) := \varinjlim_{K \Subset \Omega} \varprojlim_{h \rightarrow 0^+} \mathcal{D}_K^{M_p, h}, \quad \mathcal{D}^{\{M_p\}}(\Omega) := \varinjlim_{K \Subset \Omega} \varinjlim_{h \rightarrow \infty} \mathcal{D}_K^{M_p, h}.$$

The elements of the dual space $\mathcal{D}'^{(M_p)}(\Omega)$ are called *ultradistributions of class (M_p) (of Beurling type) in Ω* while the elements of $\mathcal{D}'^{\{M_p\}}(\Omega)$ are called *ultradistributions of class $\{M_p\}$ (of Roumieu type) in Ω* . Since multiplication and (ultra)differential operators act support shrinking and continuously on $\mathcal{E}^*(\Omega)$ (cf. Subsection 2.2.2), we can define their action on $\mathcal{D}'^*(\Omega)$ via duality. More precisely:

- Let $\psi \in \mathcal{E}^*(\Omega)$ and let $f \in \mathcal{D}'^*(\Omega)$. We define $\psi f \in \mathcal{D}'^*(\Omega)$ via $\langle \psi f, \varphi \rangle := \langle f, \psi \varphi \rangle$ for all $\varphi \in \mathcal{D}^*(\Omega)$.

- Assume that M_p satisfies $(M.2)'$. Let $f \in \mathcal{D}'^*(\Omega)$. We define $\partial_i f \in \mathcal{D}'^*(\Omega)$ via $\langle \partial_i f, \varphi \rangle := -\langle f, \partial_i \varphi \rangle$ for all $\varphi \in \mathcal{D}^*(\Omega)$.
- Assume that M_p satisfies $(M.2)$. Let $P(D)$ be an ultradifferential operator of class $*$ and let $f \in \mathcal{D}'^*(\Omega)$. We define $P(D)f \in \mathcal{D}'^*(\Omega)$ via $\langle P(D)f, \varphi \rangle := \langle f, P(-D)\varphi \rangle$ for all $\varphi \in \mathcal{D}^*(\Omega)$.

Next, we discuss the sheaf properties of \mathcal{D}'^* . Let $\Omega' \subseteq \Omega$ be an inclusion of open subsets of \mathbb{R}^d . We define the restriction mapping $\mathcal{D}'^*(\Omega) \rightarrow \mathcal{D}'^*(\Omega')$ as the transpose of the inclusion mapping $\mathcal{D}^*(\Omega') \rightarrow \mathcal{D}^*(\Omega)$, which turns \mathcal{D}'^* into a presheaf on \mathbb{R}^d . The existence of partitions of the unity of ultradifferentiable functions of class $*$ [89, Prop. 5.2] implies that \mathcal{D}'^* is a fine sheaf (cf. [89, Thm. 5.6]). The support of an element $f \in \mathcal{D}'^*(\Omega)$ coincides with the complement in Ω of the largest open set $\Omega' \subseteq \Omega$ for which it holds that $\langle f, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}^*(\Omega')$. We may identify $\mathcal{E}'^*(\Omega)$ with $\Gamma_c(\Omega, \mathcal{D}'^*)$ [89, Thm. 5.9]. Therefore the elements of $\mathcal{E}'^*(\Omega)$ are called *compactly supported ultradistributions of class $*$ in Ω* . Finally, we mention the following converse of Lemma 2.2.7.

Lemma 2.2.9. ([89, Lemma 3.3 part 1]) *Let M_p be a weight sequence satisfying $(M.1)$ and $(M.3)'$. Let $h > 0$ and let $K \Subset \mathbb{R}^d$. Then,*

$$|\widehat{\varphi}(\xi)| \leq M_0 |K| \|\varphi\|_{\mathcal{E}^{M_p, h}(K)} e^{-M(\xi/(\sqrt{d}h))}, \quad \xi \in \mathbb{R}^d,$$

for all $\varphi \in \mathcal{D}_K^{M_p, h}$.

2.2.4 Infrahyperfuctions

Local quasianalytic functionals

Let M_p be a weight sequence satisfying $(M.1)$, $(M.2)'$, and (QA) . In addition, we assume that M_p satisfies (NA) in the Beurling case and (NE) in the Roumieu case. The elements of the dual

space $\mathcal{E}'^{(M_p)}(\Omega)$ are called *quasianalytic functionals of class (M_p) (of Beurling type)* in Ω while the elements of $\mathcal{E}'^{\{M_p\}}(\Omega)$ are called *quasianalytic functionals of class $\{M_p\}$ (of Roumieu type)* in Ω . As customary, the elements of $\mathcal{A}'(\Omega)$ are called *analytic functionals* in Ω . Our conditions on M_p guarantee that the space of entire functions is dense in $\mathcal{E}^*(\Omega)$ for each open subset Ω of \mathbb{R}^d [78, Prop. 3.2]². Hence, for every inclusion $\Omega' \subseteq \Omega$ of open subsets of \mathbb{R}^d , we may identify $\mathcal{E}'^*(\Omega')$ with its image in $\mathcal{A}'(\Omega)$ under the transpose of the restriction mapping $\mathcal{A}(\Omega) \rightarrow \mathcal{E}^*(\Omega')$. We now discuss the notion of support for quasianalytic functionals. For $K \Subset \mathbb{R}^d$ we define the ensuing space of germs

$$\mathcal{E}^*[K] := \varinjlim_{K \Subset \Omega} \mathcal{E}^*(\Omega).$$

Let $\Omega \subseteq \mathbb{R}^d$ be open. Observe that we have the following isomorphism of l.c.s.

$$\mathcal{E}^*(\Omega) \cong \varprojlim_{K \Subset \Omega} \mathcal{E}^*[K].$$

Since $\mathcal{E}^*(\Omega)$ is dense in each $\mathcal{E}^*[K]$, we therefore have the algebraic isomorphism

$$\mathcal{E}'^*(\Omega) \cong \varinjlim_{K \Subset \Omega} \mathcal{E}'^*[K].$$

Let $f \in \mathcal{E}'^*(\mathbb{R}^d)$. A compact subset K of \mathbb{R}^d is said to be a **-carrier of f* if $f \in \mathcal{E}'^*[K]$. For every $f \in \mathcal{A}'(\mathbb{R}^d)$ there is a smallest compact set among the $\{p!\}$ -carriers of f , called the *support of f* and denoted by $\text{supp}_{\mathcal{A}'} f$. This essentially follows from the cohomology of the sheaf of germs of analytic functions [109]. An elementary proof based on the properties of the Poisson transform of analytic functionals is provided in [79, Sect. 9.1]. See [112] for a proof by means of the heat kernel method. Hörmander noticed that a similar result holds for quasianalytic functionals of Roumieu type [78].

²Hörmander actually only considers the Roumieu case but his proof can be adapted to cover the Beurling case as well.

More precisely, he showed that for every $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$ there is a smallest compact set among the $\{M_p\}$ -carriers of f and that this set coincides with $\text{supp}_{\mathcal{A}'} f$. The corresponding statement for the Beurling case was shown in [76]³. For future reference, we collect these facts in the following proposition.

Proposition 2.2.10. ([78, Cor. 3.5], [76, Thm. 4.11]) *Let M_p be a weight sequence satisfying (M.1), (M.2)', (QA), and (NA) ((M.1), (M.2)', (QA), and (NE)). Then, for every $f \in \mathcal{E}'^*(\mathbb{R}^d)$ the set $\text{supp}_{\mathcal{A}'} f$ is the smallest compact set among the $*$ -carriers of f .*

Sheaves of infrahyperfunctions

Let M_p be a weight sequence satisfying (M.1), (M.2)', (QA), and (NE). In [78], Hörmander constructed a flabby sheaf $\mathfrak{B}^{\{M_p\}}$ on \mathbb{R}^d such that, for all $K \Subset \mathbb{R}^d$, the space of global sections of $\mathfrak{B}^{\{M_p\}}$ with support in K is given by $\mathcal{E}'^{\{M_p\}}[K]$, that is, the space of quasi-analytic functionals of class $\{M_p\}$ supported in K . More precisely, we have that:

Proposition 2.2.11. ([78, Sect. 6]) *Let M_p be a weight sequence satisfying (M.1), (M.2)', (QA), and (NE). Then, there exists an (up to sheaf isomorphism) unique flabby sheaf $\mathfrak{B}^{\{M_p\}}$ on \mathbb{R}^d such that*

$$\Gamma_K(\mathbb{R}^d, \mathfrak{B}^{\{M_p\}}) = \mathcal{E}'^{\{M_p\}}[K]$$

for all $K \Subset \mathbb{R}^d$. Moreover, for every relatively compact open subset Ω of \mathbb{R}^d , we have that

$$\mathfrak{B}^{\{M_p\}}(\Omega) = \mathcal{E}'^{\{M_p\}}[\overline{\Omega}] / \mathcal{E}'^{\{M_p\}}[\partial\Omega].$$

We call $\mathfrak{B}^{\{M_p\}}$ the *sheaf of infrahyperfunctions of class $\{M_p\}$ (of Roumieu type)*.

³The authors work there with the notion of ultradifferentiability defined via a weight function in the sense of Braun, Meise, and Taylor [21] but their proofs can be adapted to the present setting.

Remark 2.2.12. The sheaf $\mathfrak{B}^{\{p!\}}$ coincides with the sheaf of hyperfunctions. In this case, we shall use the standard notation $\mathfrak{B} = \mathfrak{B}^{\{p!\}}$.

Our aim is to give a slightly alternative proof of Proposition 2.2.11 and discuss some of the basic properties of the sheaf $\mathfrak{B}^{\{M_p\}}$. We shall use the following general method for the construction of flabby sheaves with prescribed compactly supported sections due to Junker and Ito [84, 85], who used it to construct sheaves of vector-valued (Fourier) hyperfunctions. The idea goes back to Martineau's duality approach to hyperfunctions [109, 147].

Lemma 2.2.13. ([84, Thm. 1.2]) *Let X be a second countable, locally compact topological space. Assume that, for each compact $K \Subset X$, a Fréchet space F_K is given and that, for each inclusion $K_1 \subseteq K_2$ of compact subsets of X , there is an injective linear continuous mapping $\iota_{K_2, K_1} : F_{K_1} \rightarrow F_{K_2}$ such that $\iota_{K_3, K_1} = \iota_{K_3, K_2} \circ \iota_{K_2, K_1}$ and $\iota_{K_1, K_1} = \text{id}$ for all $K_1 \subseteq K_2 \subseteq K_3$. Suppose that the following conditions are satisfied (we identify F_{K_1} with its image in F_{K_2} under the mapping ι_{K_2, K_1}):*

(FS1) *Let $K_1 \subseteq K_2$ be an inclusion of compact subsets of X such that every connected component of K_2 meets K_1 . Then, F_{K_1} is dense in F_{K_2} .*

(FS2) *For all $K_1, K_2 \Subset X$ the mapping $F_{K_1} \times F_{K_2} \rightarrow F_{K_1 \cup K_2} : (f_1, f_2) \rightarrow f_2 - f_1$ is surjective.*

(FS3) (i) *For all $K_1, K_2 \Subset X$ it holds that $F_{K_1 \cap K_2} = F_{K_1} \cap F_{K_2}$.*

(ii) *Let $K_1 \supseteq K_2 \supseteq \dots$ be a decreasing sequence of compact sets in X and set $K = \bigcap_n K_n$. Then, $F_K = \bigcap_n F_{K_n}$.*

(FS4) $F_\emptyset = \{0\}$.

Then, there exists an (up to sheaf isomorphism) unique flabby sheaf \mathcal{F} on X such that

$$\Gamma_K(X, \mathcal{F}) = F_K$$

for all $K \in X$. Moreover, for every relatively compact open subset U of X , we have that

$$\mathcal{F}(U) = F_{\overline{U}}/F_{\partial U}.$$

Before we turn to the proof of Proposition 2.2.11, we state a support splitting theorem due to Hörmander [78] and present a short proof of it based on duality theory. This result will also be of crucial importance in Chapter 5. Let $K_1 \subseteq K_2$ be an inclusion of compact subsets of \mathbb{R}^d . We write $r_{K_1, K_2} : \mathcal{E}^*[K_2] \rightarrow \mathcal{E}^*[K_1]$ for the canonical restriction mapping. Its transpose is the inclusion mapping $\mathcal{E}'^*[K_1] \rightarrow \mathcal{E}'^*[K_2]$.

Proposition 2.2.14. (cf. [78, Thm. 5.1]) *Let $K_1, K_2 \in \mathbb{R}^d$. Then, the sequence*

$$\begin{aligned} 0 \longrightarrow \mathcal{E}'^{\{M_p\}}[K_1 \cap K_2] &\xrightarrow{S} \mathcal{E}'^{\{M_p\}}[K_1] \times \mathcal{E}'^{\{M_p\}}[K_2] \\ &\xrightarrow{T} \mathcal{E}'^{\{M_p\}}[K_1 \cup K_2] \longrightarrow 0 \end{aligned}$$

is topologically exact, where $S(f) = (f, f)$ and $T(f_1, f_2) = f_2 - f_1$. Moreover, for every bounded set $B \subset \mathcal{E}'^{\{M_p\}}[K_1 \cup K_2]$ there are bounded sets $B_j \subset \mathcal{E}'^{\{M_p\}}[K_j]$, $j = 1, 2$, such that $T(B_1, B_2) = B$.

Proof. By the open mapping theorem it suffices to show that the sequence is algebraically exact. The injectivity of S is clear while the equality $\text{Im } S = \ker T$ follows from Proposition 2.2.10. We now show that T is surjective. Since the transpose of T is given by

$$\begin{aligned} \mathcal{E}^{\{M_p\}}[K_1 \cup K_2] &\rightarrow \mathcal{E}^{\{M_p\}}[K_1] \times \mathcal{E}^{\{M_p\}}[K_2] : \\ \varphi &\rightarrow (-r_{K_1, K_1 \cup K_2}(\varphi), r_{K_2, K_1 \cup K_2}(\varphi)), \end{aligned}$$

it suffices to show that this mapping is injective and has closed range. The injectivity is obvious while it has closed range because its range coincides with the kernel of the continuous mapping

$$\begin{aligned} \mathcal{E}^{\{M_p\}}[K_1] \times \mathcal{E}^{\{M_p\}}[K_2] &\rightarrow \mathcal{E}^{\{M_p\}}[K_1 \cap K_2] : \\ (\varphi_1, \varphi_2) &\rightarrow r_{K_1 \cap K_2, K_1}(\varphi_1) + r_{K_1 \cap K_2, K_2}(\varphi_2). \end{aligned}$$

The last part follows from the general fact that for any exact sequence of Fréchet spaces

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{T} Z \longrightarrow 0,$$

with X an (FS) -space, it holds that for every bounded set $B \subset Z$ there is a bounded set $A \subset Y$ such that $T(A) = B$ [114, Lemma 26.13]. \square

Proof of Proposition 2.2.11. We employ Lemma 2.2.13 with $X = \mathbb{R}^d$. Set $F_K = \mathcal{E}'^{\{M_p\}}[K]$ and ι_{K_2, K_1} equal to the inclusion mapping $\mathcal{E}'^*[K_1] \rightarrow \mathcal{E}'^*[K_2]$. Condition $(FS1)$ is a consequence of (QA) , $(FS2)$ has been shown in Proposition 2.2.14, while $(FS3)$ and $(FS4)$ are satisfied because of Proposition 2.2.10. \square

Finally, we define some basic operations on $\mathfrak{B}^{\{M_p\}}$. To do so, we shall use the following extension principle for soft sheaves.

Lemma 2.2.15. ([91, p. 226, Lemma 2.3]) *Let X be a second countable topological space and let \mathcal{F} and \mathcal{G} be soft sheaves on X . Let $\mu_c : \Gamma_c(X, \mathcal{F}) \rightarrow \Gamma_c(X, \mathcal{G})$ be a linear mapping such that $\text{supp } \mu_c(f) \subseteq \text{supp } f$ for all $f \in \Gamma_c(X, \mathcal{F})$. Then, there is a unique sheaf morphism $\mu : \mathcal{F} \rightarrow \mathcal{G}$ such that $\mu_X(f) = \mu_c(f)$ for all $f \in \Gamma_c(X, \mathcal{F})$. If, in addition, $\text{supp } \mu_c(f) = \text{supp } f$ for all $f \in \Gamma_c(X, \mathcal{F})$, then μ is injective.*

Proposition 2.2.16. *Let M_p be a weight sequence satisfying $(M.1)$, $(M.2)'$, (QA) , and (NE) .*

- (i) The sheaf of distributions \mathcal{D}' is a subsheaf of $\mathfrak{B}^{\{M_p\}}$ and $\mathfrak{B}^{\{M_p\}}$ is a subsheaf of \mathfrak{B} . Moreover, the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}' & \longrightarrow & \mathfrak{B} \\ \downarrow & \nearrow & \\ \mathfrak{B}^{\{M_p\}} & & \end{array}$$

- (ii) Let $\Omega \subseteq \mathbb{R}^d$ be open and let $\psi \in \mathcal{E}^{\{M_p\}}(\Omega)$. Then, there is a unique mapping $\mathfrak{B}^{\{M_p\}}(\Omega) \rightarrow \mathfrak{B}^{\{M_p\}}(\Omega)$ such that its restriction to $\mathcal{E}^{\{M_p\}}(\Omega)$ coincides with the mapping $\mathcal{E}^{\{M_p\}}(\Omega) \rightarrow \mathcal{E}^{\{M_p\}}(\Omega) : f \rightarrow \psi f$.
- (iii) There is a unique sheaf morphism $\partial_i : \mathfrak{B}^{\{M_p\}} \rightarrow \mathfrak{B}^{\{M_p\}}$ such that the restriction of ∂_i to $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$ coincides with the usual action of ∂_i on $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$.
- (iv) Suppose that, in addition, M_p satisfies (M.2) and let $P(D)$ be an ultradifferential operator $P(D)$ of class $\{M_p\}$. Then, there is a sheaf morphism $P(D) : \mathfrak{B}^{\{M_p\}} \rightarrow \mathfrak{B}^{\{M_p\}}$ such that the restriction of $P(D)$ to $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$ coincides with the usual action of $P(D)$ on $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$.

Proof. (i) In view of Lemma 2.2.15, this follows from the fact that the distributional and hyperfunctional support of a distribution coincide [117, Thm. 3.9.2] and from Proposition 2.2.10.

(ii) The mapping $\mathcal{E}^{\{M_p\}}(\Omega) \rightarrow \mathcal{E}^{\{M_p\}}(\Omega) : f \rightarrow \psi f$ is support shrinking. Hence, the result follows by applying Lemma 2.2.15 to the flabby sheaf $\mathfrak{B}_{|\Omega}^{\{M_p\}}$ on Ω .

(iii) and (iv) Follows from Lemma 2.2.15 and the fact that ∂_i and $P(D)$ act support shrinking on $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$. \square

2.3 Projective and inductive spectra of locally convex spaces

The main goal of this section is to present an important result about the vanishing of the derived projective limit functor for spectra of Montel (DF) -spaces due to Vogt [159] and Wengenroth [163]. In order to be able to state this result, we must first introduce various regularity conditions for (LF) -spaces. Such conditions have their roots in Palamodov's homological theory of (LF) -spaces [127]. We shall do this in quite some detail since these concepts will also play a fundamental role in Part II, where we study the regularity properties of weighted inductive limits of smooth and ultradifferentiable functions. For a more detailed account on the projective limit functor we refer to [164, 159] while our main references for (LF) -spaces are [163, 160, 10] and [164, Chap. 6].

2.3.1 Regularity conditions for (LF) -spaces

An *inductive spectrum* \mathcal{X} of l.c.s. is a sequence $(X_N)_{N \in \mathbb{N}}$ of l.c.s. such that $X_N \subseteq X_{N+1}$ with continuous inclusion for all $N \in \mathbb{N}$. The inductive limit of the spectrum \mathcal{X} , denoted by $X = \varinjlim_N X_N$, is given by the set $X = \bigcup_N X_N$ endowed with the finest locally convex topology τ for which all the inclusion mappings $X_N \rightarrow X$ are continuous. In the sequel, we shall tacitly assume that the locally convex inductive limit topology τ on X is again Hausdorff. A l.c.s. X is called an (LB) -space if it can be written as the inductive limit of a spectrum consisting of Banach spaces. Similarly, X is called an (LF) -spaces ((LFS) -space, respectively) if it can be written as the inductive limit of a spectrum consisting of Fréchet spaces ((FS) -spaces, respectively).

Let $\mathcal{X} = (X_N)_N$ be an inductive spectrum consisting of Fréchet spaces. We shall consider the following regularity conditions on \mathcal{X} [163, 160, 10, 164]:

- \mathcal{X} is said to be *boundedly retractive* if for every bounded set B in X there is $N \in \mathbb{N}$ such that B is contained in X_N and the topologies induced on B by X and X_N coincide.
- \mathcal{X} is said to be *sequentially retractive* if for every null sequence in X there is $N \in \mathbb{N}$ such that the sequence is contained and converges to zero in X_N .
- \mathcal{X} is said to be *regular* if for every bounded set B in X there is $N \in \mathbb{N}$ such that B is contained and bounded in X_N .
- \mathcal{X} is said to be α -*regular* if for every bounded set B in X there is $N \in \mathbb{N}$ such that B is contained in X_N .
- \mathcal{X} is said to be β -*regular* if for every bounded set B in X that is contained in X_N for some $N \in \mathbb{N}$ there is $M \geq N$ such that B is bounded in X_M .
- \mathcal{X} is said to satisfy condition (wQ) if for every $N \in \mathbb{N}$ there are a neighbourhood U of 0 in X_N and $M \geq N$ such that for every $K \geq M$ and every neighbourhood W of 0 in X_M there are a neighbourhood V of 0 in X_K and $\lambda > 0$ such that $V \cap U \subseteq \lambda W$. If $(\|\cdot\|_{N,n})_n$ is a fundamental sequence of seminorms for X_N , then \mathcal{X} satisfies (wQ) if and only if

$$\forall N \exists M \geq N \exists n \forall K \geq M \forall m \exists k \exists C > 0 \forall x \in X_N : \\ \|x\|_{M,m} \leq C(\|x\|_{N,n} + \|x\|_{K,k}).$$

We have the following chain of implications (cf. [163] and the references therein)

$$\begin{aligned} \text{boundedly retractive} &\implies \text{sequentially retractive} \\ &\implies (\text{quasi-})\text{complete} \implies \text{regular} \\ &\implies \alpha\text{-regular} \ (\beta\text{-regular}) \implies (wQ). \end{aligned}$$

Finally, \mathcal{X} is said to be *boundedly stable* if for every $N \in \mathbb{N}$ and every bounded set B in X_N there is $M \geq N$ such that for every $K \geq M$ the spaces X_M and X_K induce the same topology on B . There is the following important result.

Theorem 2.3.1. ([164, Thm. 6.1]) *Let \mathcal{X} be an inductive spectrum consisting of Fréchet spaces. Then, \mathcal{X} is boundedly retractive if and only if \mathcal{X} is boundedly stable and satisfies (wQ) .*

As inductive spectra consisting of Fréchet-Montel spaces are always boundedly stable, we obtain the ensuing corollary.

Corollary 2.3.2. *Let \mathcal{X} be an inductive spectrum consisting of Fréchet-Montel spaces. Then, the following statements are equivalent:*

- (i) \mathcal{X} is boundedly retractive.
- (ii) \mathcal{X} is sequentially retractive.
- (iii) X is (quasi-)complete.
- (iv) \mathcal{X} is regular.
- (v) \mathcal{X} is α -regular (β -regular).
- (vi) \mathcal{X} satisfies (wQ) .

Remark 2.3.3. In fact, the conditions in Theorem 2.3.1 are also equivalent to the fact that \mathcal{X} is acyclic or that \mathcal{X} satisfies Retakh's condition (M) . We refer to [163] for further details.

Remark 2.3.4. Let $\mathcal{X} = (X_N)_N$ and $\mathcal{Y} = (Y_N)_N$ be two inductive spectra consisting of Fréchet spaces and let (P) be any of the properties considered above. If $\varinjlim_N X_N \cong \varinjlim_N Y_N$ topologically, then \mathcal{X} satisfies (P) if and only if \mathcal{Y} does so, as follows from Grothendieck's factorization theorem. This justifies calling an (LF) -space boundedly retractive, etc. if one (and hence all) of its defining inductive spectra has this property. We shall often do this in Part II.

2.3.2 The derived projective limit functor

A *projective spectrum* \mathcal{X} of vector spaces is a sequence $(X_N)_{N \in \mathbb{N}}$ of vector spaces together with linear linking mappings $\varrho_{N+1}^N : X_{N+1} \rightarrow X_N$ for all $N \in \mathbb{N}$. We write $\varrho_N^N = \text{id}_{X_N}$ and $\varrho_M^N = \varrho_{N+1}^N \circ \cdots \circ \varrho_M^{M-1}$ for $N \leq M$. Set

$$\text{Proj}^0 \mathcal{X} = \varprojlim_{N \in \mathbb{N}} X_N := \{(x_N)_N \in \prod X_N \mid \varrho_{N+1}^N(x_{N+1}) = x_N \text{ for all } N \in \mathbb{N}\}$$

and denote by $\varrho^M : \text{Proj}^0 \mathcal{X} \rightarrow X_M : (x_N)_N \rightarrow x_M$ the projection on the M -th component. Following Vogt [159], we define

$$\text{Proj}^1 \mathcal{X} := \prod X_N / B(\mathcal{X}),$$

where

$$B(\mathcal{X}) := \{(x_N)_N \in \prod X_N \mid \exists (y_N)_N \in \prod X_N \text{ with } x_N = y_N - \varrho_{N+1}^N(y_{N+1}) \text{ for all } N \in \mathbb{N}\}.$$

This definition coincides with the original definition of Palamodov [126] formulated in the language of homological algebra (cf. [164, Sect. 3.1]). Let

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

be an exact sequence of projective spectra, that is, a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_0 & \longrightarrow & Y_0 & \longrightarrow & Z_0 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & X_1 & \longrightarrow & Y_1 & \longrightarrow & Z_1 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

consisting of horizontal short exact sequences, and assume that $\text{Proj}^1 \mathcal{X} = 0$. Via a diagram chase one can verify that the ensuing short sequence of vector spaces

$$0 \longrightarrow \text{Proj}^0 \mathcal{X} \longrightarrow \text{Proj}^0 \mathcal{Y} \longrightarrow \text{Proj}^0 \mathcal{Z} \longrightarrow 0$$

is again exact. Two projective spectra $\mathcal{X} = (X_N, \varrho_{N+1}^N)_N$ and $\mathcal{Y} = (Y_N, \sigma_{N+1}^N)_N$ are said to be equivalent if there are increasing sequences $(l_N)_N$ and $(k_N)_N$ of natural numbers with $N \leq l_N \leq k_N \leq l_{N+1}$ and linear mappings $T_N : X_{k_N} \rightarrow Y_{l_N}$ and $S_N : Y_{l_N} \rightarrow X_{k_{N-1}}$ such that $S_N \circ T_N = \varrho_{k_N}^{k_{N-1}}$ and $T_N \circ S_{N+1} = \sigma_{l_{N+1}}^{l_N}$. Clearly, $\text{Proj}^0 \mathcal{X} \cong \text{Proj}^0 \mathcal{Y}$ if \mathcal{X} and \mathcal{Y} are equivalent projective spectra. Moreover, we have the following useful result.

Lemma 2.3.5. ([164, Prop. 3.1.7]) *Let \mathcal{X} and \mathcal{Y} be equivalent projective spectra. Then, $\text{Proj}^1 \mathcal{X} \cong \text{Proj}^1 \mathcal{Y}$.*

A *projective spectrum of l.c.s.* is a projective spectrum $\mathcal{X} = (X_N, \varrho_{N+1}^N)_N$ consisting of l.c.s. and continuous linking mappings ϱ_{N+1}^N . The projective spectrum \mathcal{X} is called *reduced* if all the projection mappings $\varrho^M : \text{Proj}^0 \mathcal{X} \rightarrow X_M$ have dense range. In such a case, the sequence $(X'_N)_N$ forms an inductive spectrum of l.c.s. since the transposes of ϱ_{N+1}^N may be considered as continuous inclusion mappings. We call the inductive spectrum $(X'_N)_N$ the *dual spectrum of \mathcal{X}* and denote it by \mathcal{X}^* .

Palamadov [126] had the beautiful idea of linking the algebraic property $\text{Proj}^1 \mathcal{X} = 0$ with the topological properties of \mathcal{X} . We shall use two results of this type in Chapter 5. Firstly, we recall the well-known abstract Mittag-Leffler lemma for Fréchet spaces.

Lemma 2.3.6. ([164, Thm. 3.2]) *Let \mathcal{X} be a reduced projective spectrum of Fréchet spaces. Then, $\text{Proj}^1 \mathcal{X} = 0$.*

For projective spectra of Montel (DF)-spaces there is the following deep result due to Vogt [159] and Wengenroth [163].

Theorem 2.3.7. ([159, Thm. 3.4], [163, Thm. 3.5]) *Let \mathcal{X} be a reduced projective spectrum of Montel (DF)-spaces. Then, the following statements are equivalent:*

- (i) $\text{Proj}^1 \mathcal{X} = 0$.
- (ii) $\text{Proj}^0 \mathcal{X}$ is ultrabornological.
- (iii) \mathcal{X}^* satisfies one of the equivalent conditions from Corollary 2.3.2.

Finally, we introduce some terminology that will be frequently used throughout this work. A l.c.s. is said to be a (PLS)-space ((PLN)-space, respectively) if it can be written as the projective limit of a reduced projective spectrum consisting of (DFS)-spaces ((DFN)-spaces, respectively). We refer to the survey article [54] for more information on (PLS)- and (PLN)-spaces.

Chapter 3

Special Colombeau algebras

3.1 Introduction

During the last 30 years, the term *Colombeau algebra* has become a collective noun for differential algebras containing the space of distributions as a linear differential subspace and the space of smooth functions as a subalgebra. A wide variety of Colombeau algebras have been constructed and studied in the literature. Of particular importance for us will be the special Colombeau algebra [30, 70], the full diffeomorphism invariant local Colombeau algebra [69], and, the more recent, functional analytic approach to Colombeau algebras [120, 119]. In Chapters 4 and 6, we shall work in setting of special Colombeau algebras; due to their simple structure this will lead to a transparent development of the theory. We will take a different point of view in Chapter 7, where we shall generalize the functional analytic approach from [120, 119] and construct diffeomorphism invariant Colombeau algebras containing spaces of ultradistributions.

The basic idea behind special Colombeau algebras is to represent distributions as sequences of smooth functions. More precisely, the space of distributions is embedded into an algebra consisting of sequences of smooth functions satisfying adequate bounds; this is

done via a suitably chosen regularization procedure. This algebra is then subjected to a quotient construction to ensure that the point-wise product of smooth functions is preserved. In this auxiliary chapter, we present a general scheme of how to construct special Colombeau algebras. The idea is to replace the pair of sheaves (\mathcal{D}', C^∞) by a rather general pair of sheaves (E, F) satisfying certain natural compatibility conditions. One should think of E as a sheaf of singular objects (generalized functions) and of F as a sheaf of regular objects (ordinary functions). The scheme outlined here will be closely followed in Chapters 4 and 6. It is important to point out that the work in this chapter shows that, in our approach, there is no structural difference between the Beurling and Roumieu case, on the one hand, and between the non-quasianalytic and the quasianalytic case, on the other hand. We were very much inspired by the general scheme of construction presented in [70, Sect. 1.3].

This chapter is organized as follows. In Section 3.2, we introduce special Colombeau algebras based on a general locally convex space. The general scheme of construction is presented in Section 3.3. It is formulated in terms of the notions defined in Section 3.2. We introduce and study algebras of generalized functions of class (M_p) and $\{M_p\}$ in Section 3.4. Most notably, we provide a “null characterization” of the space of negligible elements (cf. [70, Thm. 1.2.4]). Finally, in Section 3.5, a variant of the Schwartz impossibility result in the setting of ultradistributions and infrahyperfuctions is presented.

3.2 Special Colombeau algebras based on a locally convex space

In this section, we present a new approach to special Colombeau algebras based on a general locally convex space. Furthermore,

we discuss some of the basic properties of these algebras. Special Colombeau algebras based on a general l.c.s. have already been studied in [47] but our approach is fundamentally different.

Let F be a l.c.s. and let Λ be a sequence space, i.e. a vector subspace of $\mathbb{C}^{\mathbb{N}}$. The space Λ is said to be *normal* if for all $(x_n)_n, (y_n)_n \in \mathbb{C}^{\mathbb{N}}$ it holds that $(y_n)_n \in \Lambda$ if $(x_n)_n \in \Lambda$ and $|y_n| \leq |x_n|$ for all $n \in \mathbb{N}$. We define $\Lambda(F)$ as the set consisting of all sequences $(\varphi_n)_n \in F^{\mathbb{N}}$ such that $(q(\varphi_n))_n \in \Lambda$ for all $q \in \text{csn}(F)$. If Λ is normal, then $\Lambda(F)$ is a vector subspace of $F^{\mathbb{N}}$. A pair (Λ, Δ) of sequence spaces is called *admissible* if the following conditions are satisfied:

- Λ and Δ are normal.
- Λ is a subalgebra of $\mathbb{C}^{\mathbb{N}}$ and Δ is an ideal of Λ .
- $(1, 1, \dots) \in \Lambda$.
- $x_n \rightarrow 0$ for all $(x_n)_n \in \Delta$.

The elements of $\Lambda(F)$ and $\Delta(F)$ are called (Λ, Δ) -*moderate* and (Λ, Δ) -*negligible sequences in F* , respectively. The associated special Colombeau algebra is defined as the quotient

$$\mathcal{G}(F) = \mathcal{G}(F, \Lambda, \Delta) := \Lambda(F) / \Delta(F).$$

The equivalence class of $(\varphi_n)_n \in \Lambda(F)$ is denoted by $[(\varphi_n)_n]$.

Example 3.2.1. As customary, we denote by s the Fréchet space consisting of all sequences $(x_n)_n \in \mathbb{C}^{\mathbb{N}}$ such that $\sup_n |x_n|(1+n)^k < \infty$ for all $k \in \mathbb{N}$. Its dual s' consists of all polynomially bounded sequences. Clearly, (s', s) is admissible. The associated algebras $\mathcal{G}(F, s', s)$ have been thoroughly studied in [61]. In particular, $\mathcal{G}(C^\infty(\Omega), s', s)$ is equal to the classical special Colombeau algebra $\mathcal{G}(\Omega)$ (based on sequences instead of nets).

Example 3.2.2. Let M_p be a weight sequence. For $\lambda > 0$ we write $s^{M_p, \lambda}$ for the Banach space consisting of all sequences $(x_n)_n \in \mathbb{C}^{\mathbb{N}}$ such that $\sup_n |x_n| e^{M(\lambda n)} < \infty$. We define

$$s^{(M_p)} := \varprojlim_{\lambda \rightarrow \infty} s^{M_p, \lambda}, \quad s^{\{M_p\}} := \varinjlim_{\lambda \rightarrow 0^+} s^{M_p, \lambda}.$$

Similarly, we write $s^{M_p, -\lambda}$ for the Banach space consisting of all sequences $(x_n)_n \in \mathbb{C}^{\mathbb{N}}$ such that $\sup_n |x_n| e^{-M(\lambda n)} < \infty$. We define

$$s'^{(M_p)} := \varinjlim_{\lambda \rightarrow \infty} s^{M_p, -\lambda}, \quad s'^{\{M_p\}} := \varprojlim_{\lambda \rightarrow 0} s^{M_p, -\lambda}.$$

If M_p satisfies (M.2), then (s'^*, s^*) is admissible. These pairs of admissible sequence spaces will be used to construct algebras of generalized functions of class $*$ in Section 3.4. More precisely, we shall consider $\mathcal{G}(\mathcal{E}^*(\Omega), s'^*, s^*)$.

For later use, we remark that, if M_p satisfies (M.1) and (M.2)', Corollary 2.2.5, Theorem 9.1.3, and Remark 9.1.5¹ imply that a sequence $(x_n)_n \in \mathbb{C}^{\mathbb{N}}$ belongs to $s^{\{M_p\}}$ if and only if

$$\|(x_n)_n\|_{s^{M_p, \lambda_j}} := \sup_{n \in \mathbb{N}} |x_n| e^{M_{\lambda_j}(n)} < \infty$$

for all $\lambda_j \in \mathfrak{R}$. Moreover, the topology of $s^{\{M_p\}}$ is generated by the system of norms $\{\|\cdot\|_{s^{M_p, \lambda_j}} \mid \lambda_j \in \mathfrak{R}\}$.

The following result follows directly from our definitions. In fact, our definitions are tailor-made for these properties to hold.

Proposition 3.2.3. *Let (Λ, Δ) be an admissible pair of sequence spaces.*

(i) *The mapping*

$$\sigma : F \rightarrow \mathcal{G}(F) : \varphi \rightarrow [(\varphi)_n] \quad (3.2.1)$$

is well-defined, linear, and injective.

¹ Theorem 9.1.3 and Remark 9.1.5 will be given later on in Chapter 9.

(ii) Let $k \in \mathbb{N}$ and let F_1, \dots, F_k be locally convex spaces. Let $T : F_1 \times \dots \times F_k \rightarrow F$ be a jointly continuous multilinear mapping. Consider the mapping $\tilde{T} : F_1^{\mathbb{N}} \times \dots \times F_k^{\mathbb{N}} \rightarrow F^{\mathbb{N}}$ given by

$$\tilde{T}((\varphi_{1,n})_n, \dots, (\varphi_{k,n})_n) := (T(\varphi_{1,n}, \dots, \varphi_{k,n}))_n.$$

Then,

$$\tilde{T}(\Lambda(F_1), \dots, \Lambda(F_k)) \subseteq \Lambda(F)$$

and

$$\tilde{T}((\varphi_{1,n})_n, \dots, (\varphi_{k,n})_n) \in \Delta(F)$$

if $(\varphi_{i,n})_n \in \Lambda(F_i)$ for all $i = 1, \dots, k$ and if at least one $(\varphi_{i,n})_n$ belongs to $\Delta(F_i)$. Consequently, the mapping $\tilde{T} : \mathcal{G}(F_1) \times \dots \times \mathcal{G}(F_k) \rightarrow \mathcal{G}(F)$ given by

$$\tilde{T}([(\varphi_{1,n})_n], \dots, [(\varphi_{k,n})_n]) := [\tilde{T}((\varphi_{1,n})_n, \dots, (\varphi_{k,n})_n)]_n$$

is well-defined and commutes with the embedding σ in the sense that

$$\tilde{T}(\sigma(\varphi_1), \dots, \sigma(\varphi_k)) = \sigma(T(\varphi_1, \dots, \varphi_k)).$$

Proposition 3.2.3(ii) can be used to extend linear continuous mappings $F_1 \rightarrow F_2$ to linear mappings $\mathcal{G}(F_1) \rightarrow \mathcal{G}(F_2)$, e.g. we shall employ this result to define the action of (ultra)differential operators on our algebras of generalized functions of class $*$. Moreover, it defines an algebra structure on $\mathcal{G}(F)$ if F is a locally convex algebra. More precisely, we have that:

Corollary 3.2.4. *Let F be a locally convex algebra and let (Λ, Δ) be an admissible pair of sequence spaces. Then, $\mathcal{G}(F)$ is an algebra with multiplication given by*

$$[(\varphi_{1,n})_n] \cdot [(\varphi_{2,n})_n] := [(\varphi_{1,n} \cdot \varphi_{2,n})_n]$$

and σ is an algebra homomorphism.

Finally, we give an alternative description of the space $\Lambda(F)$ in the case that F is a projective limit of regular (LB) -spaces and Λ is of some particular form. This result shall be used to give natural representations of the spaces of $(s'^{\{M_p\}}, s^{\{M_p\}})$ -moderate and -negligible sequences in $\mathcal{E}^{\{M_p\}}(\Omega)$ in Section 3.4. We start by simply considering regular (LB) -spaces. Recall that \preceq stands for eventual domination (cf. Subsection 2.2.1).

Lemma 3.2.5. *Let $F = \varprojlim_m F_m$ be a regular (LB) -space.*

- (i) *Let $\lambda_k = (\lambda_{k,n})_n$, $k \in \mathbb{N}$, be sequences of positive reals such that $\lambda_k \preceq \lambda_{k+1}$ for all $k \in \mathbb{N}$. Set $\Lambda = \cap_k \Lambda_k$, where Λ_k denotes the space consisting of all sequences $(x_n)_n \in \mathbb{C}^{\mathbb{N}}$ such that $\sup_n |x_n| \lambda_{k,n} < \infty$. Then, $(\varphi_n)_n \in F^{\mathbb{N}}$ belongs to $\Lambda(F)$ if and only if*

$$\forall k \exists m : \sup_{n \in \mathbb{N}} \|\varphi_n\|_{F_m} \lambda_{k,n} < \infty.$$

- (ii) *Let $\lambda_k = (\lambda_{k,n})_n$, $k \in \mathbb{N}$, be sequences of positive reals such that $\lambda_{k+1} \preceq \lambda_k$ for all $k \in \mathbb{N}$. Set $\Lambda = \cup_k \Lambda_k$. Then, $(\varphi_n)_n \in F^{\mathbb{N}}$ belongs to $\Lambda(F)$ if and only if*

$$\exists k \exists m : \sup_{n \in \mathbb{N}} \|\varphi_n\|_{F_m} \lambda_{k,n} < \infty.$$

Proof. (i) We have that $(\varphi_n)_n \in \Lambda(F)$ if and only if

$$\begin{aligned} & \forall q \in \text{csn}(F) : (q(\varphi_n))_n \in \Lambda \\ & \iff \forall k \forall q \in \text{csn}(F) : \sup_{n \in \mathbb{N}} q(\varphi_n) \lambda_{k,n} < \infty \\ & \iff \forall k : \{\varphi_n \lambda_{k,n} \mid n \in \mathbb{N}\} \subset F \text{ is bounded.} \end{aligned}$$

Since F is regular, the latter is equivalent to

$$\begin{aligned} & \forall k \exists m : \{\varphi_n \lambda_{k,n} \mid n \in \mathbb{N}\} \subset F_m \text{ is bounded} \\ & \iff \forall k \exists m : \sup_{n \in \mathbb{N}} \|\varphi_n\|_{F_m} \lambda_{k,n} < \infty. \end{aligned}$$

(ii) We only need to show the direct implication, the “if” part is clear. To this end, we equip each Λ_k with the norm $\sup_n |x_n| \lambda_{k,n}$, $(x_n)_n \in \Lambda_k$. In such a way, the sequence $(\Lambda_k)_k$ becomes an inductive spectrum of Banach spaces and we endow $\Lambda = \cup_k \Lambda_k$ with the corresponding (LB) -space structure. Our assumption implies that the mapping $T : F' \rightarrow \Lambda : y' \rightarrow (\langle y', \varphi_n \rangle)_n$ is well-defined and bounded. Since F' is a Fréchet space, Grothendieck’s factorization theorem yields that $T(F') \subseteq \Lambda_k$ for some $k \in \mathbb{N}$. Hence, $\sup_n |\langle y', \varphi_n \rangle| \lambda_{k,n} < \infty$ for all $y' \in F'$, which means that the set $\{\varphi_n \lambda_{k,n} \mid n \in \mathbb{N}\}$ is weakly bounded and, thus, bounded in F . The result now follows from the fact that F is regular (cf. the last part of the proof of (i)). \square

Lemma 3.2.5 immediately yields the following corollary.

Corollary 3.2.6. *Let F be a projective limit of regular (LB) -spaces, i.e. $F = \varprojlim_M F_M$ where $(F_M)_M$ is projective spectrum of regular (LB) -spaces. Suppose that $F_M = \varinjlim_m F_{M,m}$.*

- (i) *Let $\lambda_k = (\lambda_{k,n})_n$, $k \in \mathbb{N}$, be sequences of positive reals such that $\lambda_k \preceq \lambda_{k+1}$ for all $k \in \mathbb{N}$. Set $\Lambda = \cap_k \Lambda_k$. Then, $(\varphi_n)_n \in F^\mathbb{N}$ belongs to $\Lambda(F)$ if and only if*

$$\forall M \forall k \exists m : \sup_{n \in \mathbb{N}} \|\varphi_n\|_{F_{M,m}} \lambda_{k,n} < \infty.$$

- (ii) *Let $\lambda_k = (\lambda_{k,n})_n$, $k \in \mathbb{N}$, be sequences of positive reals such that $\lambda_{k+1} \preceq \lambda_k$ for all $k \in \mathbb{N}$. Set $\Lambda = \cup_k \Lambda_k$. Then, $(\varphi_n)_n \in F^\mathbb{N}$ belongs to $\Lambda(F)$ if and only if*

$$\forall M \exists k \exists m : \sup_{n \in \mathbb{N}} \|\varphi_n\|_{F_{M,m}} \lambda_{k,n} < \infty.$$

3.3 A general scheme of construction

We now outline a general method to attack the problem of embedding linear generalized function spaces into differential algebras. We

start by giving an abstract formulation of the principal problem to be addressed in the subsequent chapters of this part.

Let (E, F) be a pair of sheaves (of vector spaces) on \mathbb{R}^d satisfying the ensuing properties:

(P1) $F \subseteq C^\infty \subseteq E$ as sheaves.

(P2) For each open set $\Omega \subseteq \mathbb{R}^d$ the space $F(\Omega)$ is a subalgebra of $C^\infty(\Omega)$ and $\partial_i(F(\Omega)) \subseteq F(\Omega)$ for $i = 1, \dots, d$.

(P3) There are sheaf morphisms $\partial_i : E \rightarrow E$ that extend the usual partial derivatives $\partial_i : C^\infty \rightarrow C^\infty$. Moreover, it holds that $\text{supp } \partial_i f \subseteq \text{supp } f$ for all $\Omega \subseteq \mathbb{R}^d$ open and $f \in E(\Omega)$.

As mentioned before, one should think of E as a sheaf of singular objects and of F as a sheaf of regular objects. The chief examples to keep in mind are $(E, F) = (\mathcal{D}', C^\infty)$, $(E, F) = (\mathcal{D}'^*, \mathcal{E}^*)$, and $(E, F) = (\mathfrak{B}^{\{M_p\}}, \mathcal{E}^{\{M_p\}})$.

In addition, the spaces $F(\Omega)$ can often be naturally endowed with a locally convex topology satisfying certain compatibility conditions with the previous properties. Hence we also introduce:

(P4) For each open set $\Omega \subseteq \mathbb{R}^d$ the space $F(\Omega)$ is endowed with a locally convex topology satisfying the ensuing properties:

- The inclusion $F(\Omega) \subseteq C^\infty(\Omega)$ is continuous.
- The restriction mappings $F(\Omega) \rightarrow F(\Omega')$ are continuous for all open subsets $\Omega' \subseteq \Omega$.
- The space $F(\Omega)$ is a locally convex algebra.
- The partial derivatives $\partial_i : F(\Omega) \rightarrow F(\Omega)$ are continuous.

The fundamental problem in the non-linear theory of generalized functions may then be formulated as follows.

Problem 3.3.1. *Let (E, F) be a pair of sheaves on \mathbb{R}^d satisfying (P1)-(P3). Construct a sheaf \mathcal{G} of algebras on \mathbb{R}^d and a sheaf monomorphism of vector spaces $\iota : E \rightarrow \mathcal{G}$ such that the following properties hold:*

- *The restriction of ι to F is a sheaf monomorphism of algebras. Explicitly, this means that $\iota_\Omega(1) = 1$ and $\iota_\Omega(\varphi\psi) = \iota_\Omega(\varphi) \cdot \iota_\Omega(\psi)$ for all open subsets $\Omega \subseteq \mathbb{R}^d$ and $\varphi, \psi \in F(\Omega)$.*
- *There are sheaf morphisms $\tilde{\partial}_i : \mathcal{G} \rightarrow \mathcal{G}$, $i = 1, \dots, d$, that commute with the embedding ι in the sense that $\iota \circ \partial_i = \tilde{\partial}_i \circ \iota$. Moreover, $\tilde{\partial}_i : \mathcal{G}(\Omega) \rightarrow \mathcal{G}(\Omega)$ is a derivation for all open subsets $\Omega \subseteq \mathbb{R}^d$.*

Problem 3.3.1 for $(E, F) = (\mathcal{D}', C^\infty)$ is the basic question in classical Colombeau theory. In Chapters 4 and 6, we shall solve Problem 3.3.1 in the following two cases:

- $(E, F) = (\mathcal{D}'^{(M_p)}, \mathcal{E}^{(M_p)})$ and $(E, F) = (\mathcal{D}'^{\{M_p\}}, \mathcal{E}^{\{M_p\}})$, where M_p is a weight sequence satisfying (M.1), (M.2), and (M.3)'.
- $(E, F) = (\mathfrak{B}^{\{M_p\}}, \mathcal{E}^{\{M_p\}})$, where M_p is a weight sequence satisfying (M.1), (M.2), (M.2)*, (QA), and (NE).

In the remainder of this section, we describe a general method to tackle Problem 3.3.1. Let (E, F) be a pair of sheaves satisfying assumptions (P1)-(P4) and let (Λ, Δ) be an admissible pair of sequence spaces. For $\Omega \subseteq \mathbb{R}^d$ open we write $\Lambda(\Omega, F) = \Lambda(F(\Omega))$, $\Delta(\Omega, F) = \Delta(F(\Omega))$, and $\mathcal{G}(\Omega, F) = \mathcal{G}(F(\Omega))$. Furthermore, we denote by $\sigma_\Omega : F(\Omega) \rightarrow \mathcal{G}(\Omega)$ the constant embedding given by (3.2.1). Corollary 3.2.4 yields that $\mathcal{G}(\Omega, F)$ is an algebra with multiplication given by $[(\varphi_{1,n})_n] \cdot [(\varphi_{2,n})_n] = [(\varphi_{1,n}\varphi_{2,n})_n]$ and that σ_Ω is an algebra monomorphism. Next, we define a differential structure on $\mathcal{G}(\Omega, F)$. By applying Proposition 3.2.3(ii) to the partial derivatives $\partial_i : F(\Omega) \rightarrow F(\Omega)$, $i = 1, \dots, d$, we obtain linear mappings $\tilde{\partial}_i : \mathcal{G}(\Omega, F) \rightarrow \mathcal{G}(\Omega, F)$ given by $\tilde{\partial}_i[(\varphi_n)_n] = [(\partial_i \varphi_n)_n]$. Obviously,

$\tilde{\partial}_i$ is a derivation on $\mathcal{G}(\Omega, F)$. We now endow $\mathcal{G}(\cdot, F)$ with a presheaf structure. Let $\Omega' \subseteq \Omega$ be an inclusion of open subsets of \mathbb{R}^d . We define a restriction mapping $\mathcal{G}(\Omega, F) \rightarrow \mathcal{G}(\Omega', F)$ by extending the continuous restriction mapping $F(\Omega) \rightarrow F(\Omega')$ via Proposition 3.2.3(ii). Explicitly, the restriction mapping $\mathcal{G}(\Omega, F) \rightarrow \mathcal{G}(\Omega', F)$ is given by $[(\varphi_n)_n]_{|\Omega'} = [(\varphi_n|_{\Omega'})_n]$. This turns $\mathcal{G}(\cdot, F)$ into a presheaf of algebras satisfying (S1) and $\sigma : F \rightarrow \mathcal{G}(\cdot, F)$ into a presheaf monomorphism of algebras. Furthermore, the partial derivatives $\tilde{\partial}_i : \mathcal{G}(\cdot, F) \rightarrow \mathcal{G}(\cdot, F)$ are presheaf morphisms such that

$$\text{supp } \tilde{\partial}_i [(\varphi_n)_n] \subseteq \text{supp } [(\varphi_n)_n]$$

for all $\Omega \subseteq \mathbb{R}^d$ open and $[(\varphi_n)_n] \in \mathcal{G}(\Omega, F)$. However, it is from the outset not at all clear whether $\mathcal{G}(\cdot, F)$ satisfies (S2). Hence, our first task is to:

(T1) *Show that $\mathcal{G}(\cdot, F)$ is a sheaf. In fact, it suffices to show that it satisfies (S2).*

Finally, we discuss the embedding of E into $\mathcal{G}(\cdot, F)$. Our goal is to construct a sheaf monomorphism $\iota : E \rightarrow \mathcal{G}(\cdot, F)$ such that $\iota|_F = \sigma$ and $\iota \circ \partial_i = \tilde{\partial}_i \circ \iota$. As σ is a sheaf monomorphism of algebras, this would solve Problem 3.3.1. To construct ι , we shall employ the extension principle for soft sheaves stated in Lemma 2.2.15. For this lemma to be applicable, the sheaves E and $\mathcal{G}(\cdot, F)$ need to be soft. Therefore, our second task becomes:

(T2) *Show that the sheaves E and $\mathcal{G}(\cdot, F)$ are soft.*

Our third task is then to:

(T3) *Construct a linear embedding*

$$\iota_c : \Gamma_c(\mathbb{R}^d, E) \rightarrow \Gamma_c(\mathbb{R}^d, \mathcal{G}(\cdot, F))$$

such that $\text{supp } \iota_c(f) = \text{supp } f$ for all $f \in \Gamma_c(\mathbb{R}^d, E)$ and $\iota_c \circ \partial_i = \tilde{\partial}_i \circ \iota_c$.

The embedding ι_c can be constructed by finding a sequence $(T_n)_n$ consisting of linear mappings $T_n : \Gamma_c(\mathbb{R}^d, E) \rightarrow F(\mathbb{R}^d)$ satisfying the ensuing properties:

- $(T_n(f))_n \in \Lambda(\mathbb{R}^d, F)$ for all $f \in \Gamma_c(\mathbb{R}^d, E)$.
- For all open subsets $\Omega \subseteq \mathbb{R}^d$ and all $f \in \Gamma_c(\mathbb{R}^d, E)$ it holds that $T_n(f)|_\Omega \rightarrow 0$ in $F(\Omega)$ implies $f|_\Omega = 0$ and $f|_\Omega = 0$ implies $(T_n(f)|_\Omega)_n \in \Delta(\Omega, F)$.
- $T_n \circ \partial_i = \partial_i \circ T_n$ for all $n \in \mathbb{N}$.

The mapping ι_c given by $\iota_c(f) := [(T_n(f))_n]$ then satisfies all requirements. We call $(T_n)_n$ a *sequence of (Λ, Δ) -regularization operators*. Lemma 2.2.15 now implies that there is a unique sheaf morphism $\iota : E \rightarrow \mathcal{G}(\cdot, F)$ such that $\iota_c(f) = \iota_{\mathbb{R}^d}(f)$ for all $f \in \Gamma_c(\mathbb{R}^d, E)$. In addition, we have that $\iota \circ \partial_i = \tilde{\partial}_i \circ \iota$, as follows from another application of Lemma 2.2.15. Our last task is to:

(T4) *Show that the restriction of ι to F coincides with σ .*

If F is soft, it suffices to check that ι_c and σ coincide on $\Gamma_c(\mathbb{R}^d, F)$. When ι_c is defined via a sequence $(T_n)_n$ of (Λ, Δ) -regularization operators, this amounts to showing that

- $(T_n(\varphi) - \varphi)_n \in \Delta(\mathbb{R}^d, F)$ for all $\varphi \in \Gamma_c(\mathbb{R}^d, F)$.

We end this section by giving two remarks.

Remark 3.3.2. Our motivation to use Lemma 2.2.15 to solve Problem 3.3.1 stems from the fact that, for all sheaves of generalized functions, the space of compactly supported sections is given by the dual of an appropriate test function space. Hence, one may try to attempt to define the sequence $(T_n)_n$ of (Λ, Δ) -regularization operators determining the embedding ι_c via the convolution with a suitable mollifier sequence belonging to the corresponding test function space. In fact, the construction of optimal mollifier sequences

may be considered as one of the major problems in Colombeau theory, especially in its branch dealing with ultradistributions. In the non-quasianalytic case, the use of Lemma 2.2.15 will spare us some work but one could use more direct arguments, based on the existence of \mathcal{D}^* -partitions of the unity, to achieve the same goal (cf. [70, Sect. 1.2.2]). Moreover, in this case, it is also possible to directly construct the embedding ι and verify by hand that $\iota \circ \partial_i = \tilde{\partial}_i \circ \iota$, in the same spirit as has been done in [45] for the classical special Colombeau algebra. Actually, we shall implicitly do this in Chapter 7. On the other hand, in the quasianalytic case, the use of Lemma 2.2.15 will be absolutely indispensable.

Remark 3.3.3. Let (E, F) be a pair of sheaves satisfying (P1)-(P4). The choice of (Λ, Δ) is of course determined by the existence of sequences of (Λ, Δ) -regularization operators and thus, in practice, by the existence of suitable mollifier sequences. This allows one to make fairly good speculations of the appropriate choice of (Λ, Δ) by studying the rate of growth of convolution averages of elements of $\Gamma_c(\mathbb{R}^d, E)$. Nonetheless, the method described in this section is rather ad-hoc. We point out three drawbacks:

- (i) The choice of (Λ, Δ) and, thus, also the definition of $\mathcal{G}(\cdot, F)$ is not uniquely determined by the pair (E, F) . Even worse, the definition of $\mathcal{G}(\cdot, F)$ does not at all depend on E .
- (ii) The embedding of E into $\mathcal{G}(\cdot, F)$ is by no means canonical.
- (iii) There is no canonical procedure to extend sheaf morphisms $E \rightarrow E$ to $\mathcal{G}(\cdot, F) \rightarrow \mathcal{G}(\cdot, F)$. In fact, whether or not one can extend a sheaf morphism $E \rightarrow E$ crucially depends on the chosen embedding ι_c .

In Chapter 7, we will present a more intrinsic method to solve Problem 3.3.1. To this end, we shall further develop the functional analytic approach to Colombeau theory initiated by Nigsch²

²In fact, Chapter 7 is based on joint work [37] with E.A. Nigsch.

in [120, 119]. Doing so, we will resolve (ii) and (iii). However, the definitions of our algebras shall still depend on an a priori chosen growth scale. In our opinion, it would be very interesting to develop a method to solve Problem 3.3.1 that entails that the definition of the algebra \mathcal{G} is uniquely determined by the given pair (E, F) ; the recent work [122] may be considered as a first step in this direction.

3.4 Algebras of generalized functions of class (M_p) and $\{M_p\}$

In this section, we employ the general theory developed in Section 3.2 to define and study the algebras $\mathcal{G}(\mathcal{E}^*(\Omega), s'^*, s^*)$ of generalized functions of class (M_p) and $\{M_p\}$. Furthermore, we provide a null characterization of the space of negligible elements. The embedding of ultradistributions and infrahyperfuctions into our algebras is postponed to Chapters 4 and 6, respectively.

Let M_p be a weight sequence satisfying (M.1) and (M.2) and let $\Omega \subseteq \mathbb{R}^d$ be open. Condition (M.1) ensures that $\mathcal{E}^*(\Omega)$ is a locally convex algebra while, as pointed out in Example 3.2.1, (M.2) implies that the pair (s'^*, s^*) is an admissible pair of sequence spaces. We define the space of *moderate sequences of ultradifferentiable functions of class $*$ in Ω* as

$$\mathcal{E}_{\mathcal{M}}^*(\Omega) := s'^*(\mathcal{E}^*(\Omega))$$

and the space of *negligible sequences of ultradifferentiable functions of class $*$ in Ω* as

$$\mathcal{E}_{\mathcal{N}}^*(\Omega) := s^*(\mathcal{E}^*(\Omega)).$$

These spaces are thus given by (in the Roumieu case we use Corollary 3.2.6)

$$\begin{aligned} \mathcal{E}_{\mathcal{M}}^{(M_p)}(\Omega) = \{(\varphi_n)_n \in \mathcal{E}^{(M_p)}(\Omega)^{\mathbb{N}} \mid \forall K \Subset \Omega \forall h > 0 \exists \lambda > 0 : \\ \sup_{n \in \mathbb{N}} \|\varphi_n\|_{\mathcal{E}^{M_p, h}(K)} e^{-M(\lambda n)} < \infty\} \end{aligned}$$

$$\mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega) = \{(\varphi_n)_n \in \mathcal{E}^{\{M_p\}}(\Omega)^{\mathbb{N}} \mid \forall K \Subset \Omega \forall \lambda > 0 \exists h > 0 : \\ \sup_{n \in \mathbb{N}} \|\varphi_n\|_{\mathcal{E}^{M_p, h}(K)} e^{-M(\lambda n)} < \infty\}$$

$$\mathcal{E}_{\mathcal{N}}^{(M_p)}(\Omega) = \{(\varphi_n)_n \in \mathcal{E}^{(M_p)}(\Omega)^{\mathbb{N}} \mid \forall K \Subset \Omega \forall h > 0 \forall \lambda > 0 : \\ \sup_{n \in \mathbb{N}} \|\varphi_n\|_{\mathcal{E}^{M_p, h}(K)} e^{M(\lambda n)} < \infty\}$$

$$\mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega) = \{(\varphi_n)_n \in \mathcal{E}^{\{M_p\}}(\Omega)^{\mathbb{N}} \mid \forall K \Subset \Omega \exists \lambda > 0 \exists h > 0 : \\ \sup_{n \in \mathbb{N}} \|\varphi_n\|_{\mathcal{E}^{M_p, h}(K)} e^{M(\lambda n)} < \infty\}.$$

The elements of the special Colombeau algebra $\mathcal{G}(\mathcal{E}^*(\Omega), s'^*, s^*)$ are called *generalized functions of class $*$ in Ω* . We shall use the shorthand notation $\mathcal{G}^*(\Omega) = \mathcal{G}(\mathcal{E}^*(\Omega), s'^*, s^*)$. Recall that this space is given by the the factor algebra

$$\mathcal{G}^*(\Omega) = \mathcal{E}_{\mathcal{M}}^*(\Omega) / \mathcal{E}_{\mathcal{N}}^*(\Omega).$$

Remark 3.4.1. These spaces should be compared with those occurring in other works dealing with constructions of generalized function algebras based on sequences or nets of ultradifferentiable functions [47, 135, 46, 8, 9, 67]. In the Beurling case, our spaces agree with those in the aforementioned works while this is not so for the Roumieu case; the difference lies in the choice and order of quantifiers. It is important to point out that this will play an essential role when embedding the spaces $\mathcal{D}'^{\{M_p\}}(\Omega)$ and $\mathfrak{B}^{\{M_p\}}(\Omega)$ into $\mathcal{G}^{\{M_p\}}(\Omega)$ and, at the same time, preserving the product of ultradifferentiable functions of class $\{M_p\}$ in Chapters 4 and 6, respectively.

The general theory developed in Section 3.2 now yields that, for each open subset $\Omega \subseteq \mathbb{R}^d$, we have that:

- The space $\mathcal{G}^*(\Omega)$ is an algebra with multiplication given by

$$[(\varphi_{1,n})_n] \cdot [(\varphi_{2,n})_n] = [(\varphi_{1,n} \varphi_{2,n})_n].$$

- The embedding

$$\sigma_\Omega : \mathcal{E}^*(\Omega) \rightarrow \mathcal{G}^*(\Omega) : \varphi \rightarrow [(\varphi)_n]$$

is an algebra monomorphism.

- The mappings $\tilde{\partial}_i : \mathcal{G}^*(\Omega) \rightarrow \mathcal{G}^*(\Omega)$, $i = 1, \dots, d$, given by

$$\tilde{\partial}_i [(\varphi_n)_n] = [(\partial_i \varphi_n)_n]$$

are derivations.

- The restriction mappings $\mathcal{G}^*(\Omega) \rightarrow \mathcal{G}^*(\Omega')$ given by

$$[(\varphi_n)_n]_{|\Omega'} = [(\varphi_n|_{\Omega'})_n]$$

turn \mathcal{G}^* into a presheaf satisfying (S1). Moreover, $\sigma : \mathcal{E}^* \rightarrow \mathcal{G}^*$ is a presheaf monomorphism of algebras and $\tilde{\partial}_i : \mathcal{G}^* \rightarrow \mathcal{G}^*$ is a presheaf morphism.

The space $\mathcal{G}^*(\Omega)$ can also be endowed with a canonical action of ultradifferential operators $P(D)$ of class $*$. In fact, since $P(D) : \mathcal{E}^*(\Omega) \rightarrow \mathcal{E}^*(\Omega)$ is a continuous linear mapping, Proposition 3.2.3(ii) implies that:

- For every ultradifferential operator $P(D)$ of class $*$ the mapping $\tilde{P}(D) : \mathcal{G}^*(\Omega) \rightarrow \mathcal{G}^*(\Omega)$ given by

$$\tilde{P}(D)[(\varphi_n)_n] = [(P(D)\varphi_n)_n]$$

is well-defined. Moreover, $\tilde{P}(D) : \mathcal{G}^* \rightarrow \mathcal{G}^*$ is a sheaf morphism.

Next, we discuss the sheaf properties of \mathcal{G}^* in some more detail. In the non-quasianalytic case, we have the ensuing important result.

Proposition 3.4.2. *Let M_p be a weight sequence satisfying (M.1), (M.2), and (M.3)'. Then, \mathcal{G}^* is a fine sheaf.*

Proof. This essentially follows from the existence of \mathcal{D}^* -partitions of the unity; we refer the reader to [70, Thm. 1.2.4] for details. \square

Let M_p be a weight sequence satisfying (M.1), (M.2), and (M.3)' and let $\Omega \subseteq \mathbb{R}^d$ be open. Then, $f \in \mathcal{G}^*(\Omega)$ belongs to $\Gamma_c(\Omega, \mathcal{G}^*)$ if and only if there are $K \Subset \Omega$ and a representative $(\varphi_n)_n$ of f such that $\text{supp } \varphi_n \subseteq K$ for all $n \in \mathbb{N}$. We call $(\varphi_n)_n$ a *compactly supported representative* of f .

In the quasianalytic case, the question whether \mathcal{G}^* is a sheaf turns out to be much more difficult. This stems from the fact that there are no non-zero compactly supported ultradifferentiable functions of class $*$. In Chapter 6, we shall show that $\mathcal{G}^{\{M_p\}}$ is indeed a sheaf if M_p satisfies (M.1), (M.2), (M.2)*, (QA), and (NE). For this, we will use the solution to the first Cousin problem for vector-valued quasianalytic functions; see Chapter 5. Moreover, we shall also show that $\mathcal{G}^{\{M_p\}}$ is soft in this case. Next, we discuss the flabbiness of \mathcal{G}^* .

Proposition 3.4.3. *Let M_p be a weight sequence satisfying (M.1), (M.2), and (NA). Then, for any open subset Ω of \mathbb{R}^d , the presheaf $\mathcal{G}_{|\Omega}^{(M_p)}$ is not flabby.*

Proof. Choose $\Omega' \subseteq \Omega$ open such that $\partial\Omega' \cap \Omega \neq \emptyset$ and fix $x_0 \in \partial\Omega' \cap \Omega$. Then, the element $[(e^{|x-x_0|^{-2}M(n)})_n] \in \mathcal{G}^{(M_p)}(\Omega')$ has no extension to any neighbourhood of x_0 . \square

Remark 3.4.4. It is interesting that the non-flabbiness of $\mathcal{G}^{(M_p)}$ does not depend on the fact whether M_p is non-quasianalytic or not. Furthermore, we speculate that a similar result holds in the Roumieu case but we are not able to show this.

Finally, we show the null characterization of the ideal $\mathcal{E}_{\mathcal{N}}^*(\Omega)$ (cf. [70, Thm. 1.2.4]). This result is a very useful tool that will be repeatedly used in the sequel.

Proposition 3.4.5. *Let M_p be a weight sequence satisfying (M.1), (M.2), and (NA) ((M.1), (M.2), and (NE)). Then, $(\varphi_n)_n \in \mathcal{E}_{\mathcal{M}}^*(\Omega)$ belongs to $\mathcal{E}_{\mathcal{N}}^*(\Omega)$ if and only if*

$$\forall K \Subset \Omega \forall \lambda > 0 (\forall K \Subset \Omega \exists \lambda > 0) : \sup_{n \in \mathbb{N}} \|\varphi_n\|_{C(K)} e^{M(\lambda n)} < \infty.$$

The proof of Proposition 3.4.5 is based on the following (weak) multivariate version of Gorny's inequality.

Lemma 3.4.6. *Let K and K' be regular compact subsets of \mathbb{R}^d such that $K' \Subset \text{int } K$ and set $d(K', K^c) = \delta > 0$. For all $k, m \in \mathbb{N}$ with $0 < m < k$ it holds that*

$$\begin{aligned} \max_{|\alpha|=m} \|\partial^\alpha \varphi\|_{C(K')} &\leq 4e^{2m} \left(\frac{k}{m} \right)^m. \\ \|\varphi\|_{C(K)}^{1-m/k} &\left(\max \left\{ d^k \max_{|\alpha|=k} \|\partial^\alpha \varphi\|_{C(K)}, \frac{\|\varphi\|_{C(K)} k!}{\delta^k} \right\} \right)^{m/k} \end{aligned}$$

for all $\varphi \in C^k(K)$.

Proof. The proof is based on a result about bounds for directional derivatives [26] and the one-dimensional inequality of Gorny [66, p. 324]. The latter states that, for $a, b \in \mathbb{R}$ with $a < b$,

$$\begin{aligned} \|\psi^{(m)}\|_{C([a,b])} &\leq 4e^{2m} \left(\frac{k}{m} \right)^m. \\ \|\psi\|_{C([a,b])}^{1-m/k} &\left(\max \left\{ \|\psi^{(k)}\|_{C([a,b])}, \frac{\|\psi\|_{C([a,b])} 2^k k!}{(b-a)^k} \right\} \right)^{m/k} \end{aligned}$$

for all $\psi \in C^k([a, b])$ and $0 < m < k$. Denote by $\partial/\partial\xi$ the directional derivative in the direction ξ , where $\xi \in \mathbb{R}^d$ is a unit vector. In [26, Thm. 2.2], it is shown that

$$\max_{|\alpha|=m} \|\partial^\alpha \varphi\|_{C(K')} \leq \sup_{|\xi|=1} \left\| \frac{\partial^m \varphi}{\partial^m \xi} \right\|_{C(K')} \quad (3.4.1)$$

for all $\varphi \in C^m(K')$; it is for this inequality that we need the compact set K' to be regular. For $x \in K'$ we write $l(x, \xi)$ for the line in \mathbb{R}^d

with direction ξ passing through the point x . Define $\psi_{x,\xi}(t) = \varphi(x + t\xi)$ for $t \in \{t \in \mathbb{R} \mid x + t\xi \in K\}$. The latter set always contains a compact interval $I_{x,\xi} \ni 0$ of length at least 2δ . The one-dimensional inequality of Gorny now implies that, for each unit vector $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\left\| \frac{\partial^m \varphi}{\partial^m \xi} \right\|_{C(K')} &= \max_{x \in K'} |\psi_{x,\xi}^{(m)}(0)| \leq \max_{x \in K'} \|\psi_{x,\xi}^{(m)}\|_{C(I_{x,\xi})} \\
&\leq \max_{x \in K'} 4e^{2m} \left(\frac{k}{m} \right)^m \|\psi_{x,\xi}\|_{C(I_{x,\xi})}^{1-m/k} \times \\
&\quad \left(\max \left\{ \|\psi_{x,\xi}^{(k)}\|_{C(I_{x,\xi})}, \frac{\|\psi_{x,\xi}\|_{C(I_{x,\xi})} k!}{\delta^k} \right\} \right)^{m/k} \\
&\leq 4e^{2m} \left(\frac{k}{m} \right)^m \|\varphi\|_{C(K)}^{1-m/k} \left(\max \left\{ \left\| \frac{\partial^k \varphi}{\partial^k \xi} \right\|_{C(K)}, \frac{\|\varphi\|_{C(K)} k!}{\delta^k} \right\} \right)^{m/k}.
\end{aligned}$$

The result now follows from (3.4.1) and the fact that

$$\begin{aligned}
\left\| \frac{\partial^k \varphi}{\partial^k \xi} \right\|_{C(K)} &= \max_{x \in K} \left| \sum_{i_1=1}^d \cdots \sum_{i_k=1}^d \frac{\partial^k \varphi(x)}{\partial x_{i_1} \cdots \partial x_{i_k}} \xi_{i_1} \cdots \xi_{i_k} \right| \\
&\leq d^k \max_{|\alpha|=k} \|\partial^\alpha \varphi\|_{C(K)}.
\end{aligned}$$

□

Proof of Proposition 3.4.5. The direct implication is clear. We now show the “if” part. Let K' be an arbitrary regular compact set of Ω . Choose a regular compact K of Ω such that $K' \Subset \text{int } K$ and set $d(K', K^c) = \delta > 0$. We have that

$$\begin{aligned}
&\forall h_1 > 0 \exists \lambda_1 > 0 \exists C_1 > 0 (\forall \lambda_1 > 0 \exists h_1 > 0 \exists C_1 > 0) : \\
&\quad \|\partial^\alpha \varphi_n\|_{C(K)} \leq C_1 h_1^{|\alpha|} M_\alpha e^{M(\lambda_1 n)}
\end{aligned}$$

for all $\alpha \in \mathbb{N}^d$ and $n \in \mathbb{N}$, and

$$\forall \lambda_2 > 0 \exists C_2 > 0 (\exists \lambda_2 > 0 \exists C_2 > 0) : \|\varphi_n\|_{C(K)} \leq C_2 e^{-M(\lambda_2 n)}$$

for all $n \in \mathbb{N}$. Let $\beta \in \mathbb{N}^d$, $\beta \neq 0$, be arbitrary. Lemma 3.4.6 with $m = |\beta|$ and $k = 2|\beta|$ implies that

$$\|\partial^\beta \varphi_n\|_{C(K')} \leq 4(2e^2)^{|\beta|} \|\varphi_n\|_{C(K)}^{1/2} \cdot \left(\max \left\{ d^{2|\beta|} \max_{|\alpha|=2|\beta|} \|\partial^\alpha \varphi_n\|_{C(K)}, \frac{\|\varphi_n\|_{C(K)} (2|\beta|)!}{\delta^{2|\beta|}} \right\} \right)^{1/2}.$$

The result now follows by plugging in the above inequalities into the right-hand side of the latter one. \square

3.5 Impossibility results

Let M_p be a weight sequence. Our aim is to show an analogue of Schwartz's impossibility result for ultradistributions of class $*$ and infrahyperfuctions of class $\{M_p\}$. We refer to the introduction of this part (Chapter 1) for the statement of the original impossibility result of Schwartz. The role of the continuous functions in our impossibility result is played by a space of ultradifferentiable functions of class \star with slightly less regularity than those of class $*$. Moreover, in this context, it seems natural to consider algebras that are endowed with an action of ultradifferential operators $P(D)$ of class $*$. Of course, the action of $P(D)$ should extend the usual action of $P(D)$ and satisfy some version of Leibniz's rule. For the latter, we observe that

$$P(D)(qf) = \sum_{\beta \leq \deg q} \frac{1}{\beta!} D^\beta q \cdot (D^\beta P)(D)f$$

holds for all polynomials q and all ultradistributions and infrahyperfuctions f .

We now proceed with giving the precise statement of our impossibility result. Let $\Omega \subseteq \mathbb{R}^d$ be open and let X denote either $\mathcal{D}'^{(M_p)}(\Omega)$, $\mathcal{D}'^{\{M_p\}}(\Omega)$, or $\mathfrak{B}^{\{M_p\}}(\Omega)$. In the first two cases, we assume that M_p satisfies (M.1), (M.2), and (M.3)' while we assume that M_p satisfies (M.1), (M.2), (QA), and (NE) in the third case.

Let N_p be another weight sequence. Recall that $*$ stands for (M_p) or $\{M_p\}$. In addition, we write \star for (N_p) or $\{N_p\}$. When embedding X into some algebra $\mathcal{A}^{*,\star} = (\mathcal{A}^{*,\star}, +, \circ)$, the following requirements appear to be natural:

- (i) X is linearly embedded into $\mathcal{A}^{*,\star}$ and the constant function 1 is the unity in $\mathcal{A}^{*,\star}$.
- (ii) For each ultradifferential operator $P(D)$ of class $*$ there is a linear operator $P(D) : \mathcal{A}^{*,\star} \rightarrow \mathcal{A}^{*,\star}$ that satisfies the ensuing Leibniz rule

$$P(D)(q \circ f) = \sum_{\beta \leq \deg q} \frac{1}{\beta!} D^\beta q \circ (D^\beta P)(D)f, \quad f \in \mathcal{A}^{*,\star},$$

for each polynomial q . Moreover, $P(D)$ coincides with the usual action of $P(D)$ on X .

- (iii) $\circ|_{\mathcal{E}^\star(\Omega) \times \mathcal{E}^\star(\Omega)}$ coincides with the pointwise product of functions.

The next result imposes a limitation on the possibility of constructing such algebras.

Theorem 3.5.1. *Let $\Omega \subseteq \mathbb{R}^d$ be open.*

- (i) *Let M_p be a weight sequence satisfying (M.1), (M.2), and (M.3)' and set $X = \mathcal{D}'^\star(\Omega)$.*

- *There is no algebra $\mathcal{A}^{(M_p), \{M_p\}}$ satisfying conditions (i)-(iii).*
- *Let N_p be a weight sequence satisfying (M.1) and $M_p \prec N_p$. Then, there is no algebra $\mathcal{A}^{\{M_p\}, (N_p)}$ satisfying conditions (i)-(iii).*

- (ii) *Let M_p be a weight sequence satisfying (M.1), (M.2), (QA), and (NE) and set $X = \mathfrak{B}^{\{M_p\}}(\Omega)$. Let N_p be a weight sequence satisfying (M.1) and $M_p \prec N_p$. Then, there is no algebra $\mathcal{A}^{\{M_p\}, (N_p)}$ satisfying conditions (i)-(iii).*

The proof of Theorem 3.5.1 is based on the ensuing two structural theorems due to Takiguchi [154, 155]. The first one is a refinement of Komatsu's second structural theorem [89, Thm. 10.3].

Proposition 3.5.2. ([154, Thm. 5.1]) *Let M_p be a weight sequence satisfying (M.1), (M.2), and (M.3)'.*

- (i) *For every $f \in \mathcal{E}'^{(M_p)}(\mathbb{R}^d)$ there are an ultradifferential operator $P(D)$ of class (M_p) and $g \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$ such that $f = P(D)g$.*
- (ii) *Let N_p be a weight sequence satisfying (M.1) such that $M_p \prec N_p$. Then, for every $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$ there are an ultradifferential operator $P(D)$ of class $\{M_p\}$ and $g \in \mathcal{E}^{(N_p)}(\mathbb{R}^d)$ such that $f = P(D)g$.*

Proposition 3.5.3. (cf. [155, Thm. 4.1]) *Let M_p be a weight sequence satisfying (M.1), (M.2), (QA), and (NE) and let N_p be a weight sequence satisfying (M.1) such that $M_p \prec N_p$. Then, for every $f \in \mathcal{S}'(\mathbb{R}^d)$ there are an ultradifferential operator $P(D)$ of class $\{M_p\}$ and $g \in \mathcal{E}^{(N_p)}(\mathbb{R}^d)$ such that $f = P(D)g$ in $\mathfrak{B}^{\{M_p\}}(\mathbb{R}^d)$.*

Remark 3.5.4. Takiguchi actually claims that the above structural theorem holds for *all* $f \in \mathfrak{B}^{\{M_p\}}(\mathbb{R}^d)$. In this general case, the proof of [155, Thm. 4.1] is rather ambiguous and unclear. However, his technique can be used to prove the weaker statement presented in Proposition 3.5.3 and this will suffice for our purposes.

Proof of Theorem 3.5.1. (i) Suppose $\mathcal{A}^{*,*}$ is such an algebra (we treat both cases at the same time). One can employ Schwartz's original idea by making use of the following observation: $q \circ P(D)g = qP(D)g$ for all ultradifferential operators $P(D)$ of class $*$, $g \in \mathcal{E}^*(\Omega)$, and polynomials q ; this follows from conditions (i) and (ii) by using induction on the degree of q . For simplicity, we assume that $0 \in \Omega$. We write $H(x) := H(x_1) \otimes \cdots \otimes H(x_d)$, where $H(x_j)$ stands for the Heaviside function, and $\text{p.v.}(x^{-1}) :=$

$\text{p. v.}(x_1^{-1}) \otimes \cdots \otimes \text{p. v.}(x_d^{-1})$, where $\text{p. v.}(x_j^{-1})$ stands for the principal value regularization of the function x_j^{-1} . Let f_j denote either $H(x_j)$ or $\text{p. v.}(x_j^{-1})$ and let Ω_j be the projection of Ω onto the x_j -axis. By using cut-off functions, we can decompose f_j as $f_j = f_{j,1} + g_{j,2}$, where $f_{j,1} \in \mathcal{E}'^*(\Omega_j)$ and $g_{j,2} \in \mathcal{E}^*(\Omega_j)$. By Proposition 3.5.2 there are an ultradifferential operator $P_{j,1}(D_j)$ of class $*$ and $g_{j,1} \in \mathcal{E}^*(\Omega_j)$ such that $P_{j,1}(D_j)g_{j,1} = f_{j,1}$. Hence, $f_j = P_{j,1}(D_j)g_{j,1} + P_{j,2}(D_j)g_{j,2}$, where $P_{j,2}(D_j)$ is the trivial differential operator $P_{j,2}(D_j) = 1$. We conclude that H and $\text{p. v.}(x^{-1})$ are linear combinations of ultradistributions of class $*$ of the form

$$\begin{aligned} & P_{1,k_1}(D_1)g_{1,k_1} \otimes \cdots \otimes P_{d,k_d}(D_d)g_{d,k_d} \\ &= P_{1,k_1}(D_1) \cdots P_{d,k_d}(D_d)(g_{1,k_1}) \otimes \cdots \otimes g_{d,k_d}, \end{aligned}$$

where $k_j \in \{1, 2\}$ for $j = 1, \dots, d$. Therefore, H and $\text{p. v.}(x^{-1})$ can be written as linear combinations of terms of the form $P(D)g$, where $P(D)$ is an ultradifferential operator of class $*$ and $g \in \mathcal{E}^*(\Omega)$. Set $\boldsymbol{\partial} = \partial^d/\partial_1 \cdots \partial_d$ and $q(x) = x_1 \cdots x_d$. The observation made at beginning of the proof now yields that $q \circ \boldsymbol{\partial}H = (q\boldsymbol{\partial}H)$ and $q \circ \text{p. v.}(x^{-1}) = q \text{p. v.}(x^{-1})$. Since $q\boldsymbol{\partial}H = 0$ and $q \text{p. v.}(x^{-1}) = 1$ in $\mathcal{D}'^*(\Omega)$, we obtain that

$$\delta = \boldsymbol{\partial}H = \boldsymbol{\partial}H \circ (q \circ \text{p. v.}(x^{-1})) = (\boldsymbol{\partial}H \circ q) \circ \text{p. v.}(x^{-1}) = 0,$$

which contradicts the injectivity of $\mathcal{D}'^*(\Omega) \rightarrow \mathcal{A}^{*,\dagger}$.

(ii) Observe that $H(x)$ and $\text{p. v.}(x^{-1})$ belong to $\mathcal{S}'(\mathbb{R}^d)$. Hence, we can use a similar argument as in (i) by using Proposition 3.5.3 instead of Proposition 3.5.2. In fact, due to the global nature of Proposition 3.5.3, the proof becomes a bit easier. \square

Remark 3.5.5. In the non-quasianalytic case, one could also use a global structural theorem to give a slightly simpler proof of Theorem 3.5.1(i).

Despite this impossibility theorem, we shall show in Chapter 4 and 6 that the algebras $\mathcal{G}^*(\Omega)$ do satisfy the enjoyable properties (i)-(iii) for $\star = *$.

Chapter 4

Optimal embeddings of spaces of ultradistributions into differential algebras

4.1 Introduction

Let M_p be a weight sequence satisfying (M.1), (M.2), and (M.3)'. Colombeau algebras containing the space of ultradistributions of class $*$ have been defined and studied by several authors [47, 135, 46, 8, 9, 67]. However, in any of the algebras constructed so far, there has always been the rather unnatural restriction that the multiplication is preserved only for some ultradifferentiable functions that are strictly more regular than those of class $*$. The goal of this chapter is to develop a non-linear theory of ultradistributions that avoids this loss of regularity phenomenon. More precisely, we shall show that it is possible to embed the space of ultradistributions of class $*$ into the algebra of generalized functions of class $*$ in such a way that the multiplication of ultradifferentiable functions of class $*$ is preserved. In view of the impossibility result presented in Section 3.5, such an embedding is optimal. The main novelty in this chapter is the construction of suitable mollifier sequences.

Furthermore, by modifying the definition of Oberguggenberger's algebra \mathcal{G}^∞ [123], we introduce a notion of regularity of class $*$ in our algebras of generalized functions and show that it coincides with ultradifferentiability of class $*$ when restricted to ultradistributions of class $*$. Based upon this result, we then develop the basics of microlocal analysis in our algebras. In the context of classical Colombeau algebras, (micro)local analysis is a well-studied topic [35, 118, 80, 136] and provides a framework for the study of the propagation of singularities under the action of differential operators with singular coefficients [81].

This chapter is organized as follows. In Section 4.2, we follow the method presented in Section 3.3 to embed the sheaf of ultradistributions of class $*$ into the sheaf of algebras of generalized functions of class $*$ and to show that the multiplication of ultradifferentiable functions of class $*$ is preserved under this embedding. The next two sections are devoted to (micro)local analysis in our algebras. In Section 4.3, we introduce the algebra of regular generalized functions of class $*$ and show that an ultradistribution of class $*$ is a regular generalized function of class $*$ if and only if it is an ultradifferentiable function of class $*$. We define the wave front set of a generalized function of class $*$ in Section 4.4 and show that this microlocal notion is consistent with the one for ultradistributions of class $*$ [93, 133].

Unless explicitly stated otherwise, M_p will stand for a weight sequence satisfying (M.1), (M.2), and (M.3)' throughout this chapter.

4.2 Construction of the embedding

In this section, we solve Problem 3.3.1 for the pair of sheaves $(E, F) = (\mathcal{D}'^*, \mathcal{E}^*)$. We prepare the ground with a discussion about

the mollifier sequences to be employed in our embedding of $\mathcal{E}'^*(\mathbb{R}^d)$ into $\Gamma_c(\mathbb{R}^d, \mathcal{G}^*)$.

Let N_p be a weight sequence satisfying (M.1) and (M.3)' and let $\rho \in L^1(\mathbb{R}^d)$. The sequence $(\rho_n)_n$ with $\rho_n = n^d \rho(n \cdot)$ is said to be an (N_p) -mollifier sequence if

- $\widehat{\rho}$ is even.
- $\widehat{\rho} \in \mathcal{D}^{(N_p)}(\mathbb{R}^d)$.
- $\widehat{\rho}(\xi) = 1$ for $|\xi| \leq 1$.
- $\text{supp } \widehat{\rho} \subseteq \overline{B}(0, 2)$.

The following lemma is of crucial importance; it essentially follows from some results of Roumieu [141].

Lemma 4.2.1. *For every weight sequence M_p satisfying (M.1) and (M.3)' there is a weight sequence N_p satisfying (M.1), (M.3)', and the ensuing condition: For each $\lambda > 0$ there is $C > 0$ such that*

$$2M(t) \leq N(\lambda t) + C, \quad t \geq 0. \quad (4.2.1)$$

Proof. Set $Q_p = \min_{q \leq p} M_q M_{p-q}$, $p \in \mathbb{N}$. Then, the sequence Q_p satisfies (M.1) and the associated function of Q_p is equal to $2M$ [89, Lemma 3.5]. In particular, Q_p satisfies (M.3)'. By [89, Lemma 4.3] there is a weight sequence N_p satisfying (M.1), (M.3)' such that $N_p \prec Q_p$, which in turn yields (4.2.1). \square

Throughout the rest of this section we fix an (N_p) -mollifier sequence $(\rho_n)_n$, where N_p is a weight sequence satisfying the conditions from Lemma 4.2.1. We are ready to embed $\mathcal{E}'^*(\mathbb{R}^d)$ into $\Gamma_c(\mathbb{R}^d, \mathcal{G}^*)$. Notice that, since ρ_n is an entire function, the convolution $f * \rho_n = \langle f(x), \rho_n(\cdot - x) \rangle$ is well-defined for $f \in \mathcal{E}'^*(\mathbb{R}^d)$.

Proposition 4.2.2. *The mapping*

$$\iota_c : \mathcal{E}'^*(\mathbb{R}^d) \rightarrow \Gamma_c(\mathbb{R}^d, \mathcal{G}^*) : f \rightarrow [(f * \rho_n)_n]$$

is a linear embedding such that $\text{supp } \iota_c(f) = \text{supp } f$ for all $f \in \mathcal{E}'^*(\mathbb{R}^d)$ and $\tilde{P}(D) \circ \iota_c = \iota_c \circ P(D)$ for all ultradifferential operators $P(D)$ of class $*$. Moreover, the restriction of ι_c to $\mathcal{D}^*(\mathbb{R}^d)$ coincides with the constant embedding $\sigma_{\mathbb{R}^d}$.

In the proof of Proposition 4.2.2, we shall employ the following lemma.

Lemma 4.2.3. *Let N_p be a weight sequence satisfying (M.1) and (M.3)' and let $K \Subset \mathbb{R}^d$. Choose $R \geq 1$ such that $K \subseteq \overline{B}(0, R)$. Then, for every $h > 0$ there is $C > 0$ such that*

$$|\partial^\alpha \mathcal{F}(\varphi)(\xi)| \leq C(2R)^{|\alpha|} \|\varphi\|_{\mathcal{E}^{N_p, h}(K)} e^{-N(|\xi|/(2\sqrt{d}h))}, \quad \xi \in \mathbb{R}^d,$$

for all $\alpha \in \mathbb{N}^d$ and $\varphi \in \mathcal{D}_K^{N_p, h}$.

Proof. By Lemma 2.2.9 we have that

$$|\partial^\alpha \mathcal{F}(\varphi)(\xi)| \leq N_0 |K| \|x^\alpha \varphi\|_{\mathcal{E}^{N_p, 2h}(K)} e^{-N(|\xi|/(2\sqrt{d}h))}.$$

Condition (M.1) implies that

$$\|x^\alpha \varphi\|_{\mathcal{E}^{N_p, 2h}(K)} \leq \|x^\alpha\|_{\mathcal{E}^{\{N_p\}, h}(K)} \|\varphi\|_{\mathcal{E}^{\{N_p\}, h}(K)}.$$

The result now follows from the fact that $p! \prec N_p$ and, thus,

$$\begin{aligned} \|x^\alpha\|_{\mathcal{E}^{N_p, h}(K)} &= \max_{\beta \leq \alpha} \max_{x \in K} \binom{\alpha}{\beta} \frac{\beta! |x^{\alpha-\beta}|}{h^{|\beta|} N_\beta} \\ &\leq (2R)^{|\alpha|} \max_{\beta \leq \alpha} \frac{|\beta|!}{h^{|\beta|} N_\beta} \\ &\leq C(2R)^{|\alpha|}, \end{aligned}$$

for some $C > 0$. □

Proof of Proposition 4.2.2. The assertion concerning the ultradifferential operators of class $*$ is clear from the definition of ι_c . The rest of the proof is divided into four steps.

STEP I: $(f * \rho_n)_n \in \mathcal{E}_{\mathcal{M}}^*(\mathbb{R}^d)$ for all $f \in \mathcal{E}'^*(\mathbb{R}^d)$. Let $K \Subset \mathbb{R}^d$ and $h > 0$ be arbitrary. The continuity of f implies that (cf. the proof of [89, Thm. 6.1])

$$\exists K' \Subset \Omega \exists k > 0 \exists C > 0 (\exists K' \Subset \Omega \forall k > 0 \exists C > 0) :$$

$$\|f * \rho_n\|_{\mathcal{E}^{M_p, h}(K)} \leq C \|\rho_n\|_{\mathcal{E}^{M_p, l}(K-K')},$$

where $l = \min\{h, k\}/H$. By Lemma 2.2.7 we have that

$$\begin{aligned} \|\rho_n\|_{\mathcal{E}^{M_p, l}(K-K')} &\leq \frac{1}{M_0(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{\rho}(\xi/n)| e^{M(\xi/l)} d\xi \\ &\leq \frac{n^d}{M_0(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{\rho}(\xi)| e^{M(n\xi/l)} d\xi \\ &\leq \frac{n^d}{M_0(2\pi)^d} \|\widehat{\rho}\|_{L^1} e^{M(2n/l)} \\ &\leq \frac{C_0}{M_0(2\pi)^d} \|\widehat{\rho}\|_{L^1} e^{M(2Hn/l)}, \end{aligned}$$

for n large enough.

STEP II: $\text{supp } \iota_c(f) \subseteq \text{supp } f$ for all $f \in \mathcal{E}'^*(\mathbb{R}^d)$. We use Proposition 3.4.5 to show that $\iota_c(f) = 0$ on $\mathbb{R}^d \setminus \text{supp } f$. Let $K \Subset \mathbb{R}^d \setminus \text{supp } f$ be arbitrary. Choose $K' \Subset \mathbb{R}^d$ such that $\text{supp } f \subset \text{int } K'$ and $K' \cap K = \emptyset$. Set $d(K, K') = c > 0$. Hence,

$$\max_{x \in K} |(f * \rho_n)(x)| \leq C' \|\rho_n\|_{\mathcal{E}^{M_p, h}(K-K')}$$

for some $C', h > 0$ (for every $h > 0$ and some $C' > 0$). Set $\varphi = \widehat{\rho} \in \mathcal{D}^{(N_p)}(\mathbb{R}^d)$ and, thus, $\rho = (2\pi)^{-d} \widehat{\varphi}(-\cdot)$. Lemma 4.2.3 implies that

$$\begin{aligned} \|\rho_n\|_{\mathcal{E}^{M_p, h}(K-K')} &= \frac{n^d}{(2\pi)^d} \max_{x \in K-K'} \sup_{\alpha \in \mathbb{N}^d} \frac{|\partial^\alpha \mathcal{F}(\varphi)(-nx)|}{(h/n)^{|\alpha|} M_\alpha} \\ &\leq \frac{C n^d}{(2\pi)^d} \|\varphi\|_{\mathcal{E}^{N_p, h/2}(\overline{B}(0,2))} e^{-N(cn/(\sqrt{d}h)) + M(4n/h)} \\ &\leq \frac{C_0 C}{(2\pi)^d} \|\varphi\|_{\mathcal{E}^{N_p, h/2}(\overline{B}(0,2))} e^{-N(cn/(\sqrt{d}h)) + M(4Hn/h)}, \end{aligned}$$

for n large enough, whence the result follows from (4.2.1).

STEP III: $\text{supp } f \subseteq \text{supp } \iota_c(f)$ for all $f \in \mathcal{E}'^*(\mathbb{R}^d)$. We have that $(f * \rho_n)_n \in \mathcal{E}'^*(\mathbb{R}^d \setminus \text{supp } \iota_c(f))$. In particular, $f * \rho_n \rightarrow 0$ uniformly on compacts of $\mathbb{R}^d \setminus \text{supp } \iota_c(f)$. Hence,

$$\langle f, \varphi \rangle = \lim_{n \rightarrow \infty} \langle f, \varphi * \rho_n \rangle = \lim_{n \rightarrow \infty} \langle f * \rho_n, \varphi \rangle = 0$$

for all $\varphi \in \mathcal{D}^*(\mathbb{R}^d \setminus \text{supp } \iota_c(f))$.

STEP IV: $\iota_c(\varphi) = \sigma_{\mathbb{R}^d}(\varphi)$ for all $\varphi \in \mathcal{D}^*(\mathbb{R}^d)$. Set $\text{supp } \varphi = K$. By Proposition 3.4.5 it suffices to show that

$$\max_{n \in \mathbb{N}} \max_{x \in K} |(\varphi * \rho_n)(x) - \varphi(x)| e^{M(\lambda n)} < \infty$$

for all $\lambda > 0$ (for some $\lambda > 0$). Lemma 2.2.9 implies that for all $h > 0$ (some $h > 0$)

$$\begin{aligned} |(\varphi * \rho_n)(x) - \varphi(x)| &= |\mathcal{F}^{-1}(\widehat{\varphi} \widehat{\rho}_n - \widehat{\varphi})(x)| \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi)| |\widehat{\rho}(\xi/n) - 1| d\xi \\ &\leq \frac{M_0 |K|}{(2\pi)^d} \|\varphi\|_{\mathcal{E}^{M_p, h}(K)} (\|\widehat{\rho}\|_{L^\infty} + 1) \int_{|\xi| \geq n} e^{-M(\xi/(\sqrt{d}h))} d\xi. \end{aligned}$$

Hence,

$$|(\varphi * \rho_n)(x) - \varphi(x)| \leq C \|\varphi\|_{\mathcal{E}^{M_p, h}(K)} e^{-M(n/(\sqrt{d}hH))}, \quad (4.2.2)$$

where

$$C = \frac{C_0 M_0 |K|}{(2\pi)^d} (\|\widehat{\rho}\|_{L^\infty} + 1) \int_{\mathbb{R}^d} e^{-M(\xi/(\sqrt{d}hH))} d\xi < \infty.$$

□

Sinc both the sheaves \mathcal{D}^* and \mathcal{G}^* are fine and, thus, soft (Subsection 2.2.3 and Proposition 3.4.2), Lemma 2.2.15 yields the ensuing result (cf. Section 3.3).

Theorem 4.2.4. *There is a unique sheaf monomorphism $\iota : \mathcal{D}'^* \rightarrow \mathcal{G}^*$ such that:*

- (i) *The restriction of $\iota_{\mathbb{R}^d}$ to $\mathcal{E}'^*(\mathbb{R}^d)$ coincides with ι_c .*
- (ii) *The restriction of ι to \mathcal{E}^* coincides with σ . In particular, $\iota_\Omega(\varphi\psi) = \iota_\Omega(\varphi) \cdot \iota_\Omega(\psi)$ for all $\Omega \subseteq \mathbb{R}^d$ open and all $\varphi, \psi \in \mathcal{E}^*(\Omega)$.*
- (iii) *For each ultradifferential operator $P(D)$ of class $*$ we have that $\tilde{P}(D) \circ \iota = \iota \circ P(D)$.*

For later use, we point out that the following consequence of inequality (4.2.2).

Lemma 4.2.5. *Let $f \in \mathcal{E}'^*(\mathbb{R}^d)$ and let $(\varphi_n)_n$ be a representative of $\iota_c(f)$. Then,*

$$\exists h > 0 \forall K \Subset \mathbb{R}^d \exists C > 0 \forall \psi \in \mathcal{D}_K^{(M_p)} \forall n \in \mathbb{N} :$$

$$|\langle f - \varphi_n, \psi \rangle| \leq C \|\psi\|_{\mathcal{E}_{M_p, h/H}(K)} e^{-M(n/(\sqrt{d}Hh))}.$$

Proof. Since the assertion is independent of the representative of $\iota_c(f)$, we may assume that $\varphi_n = f * \rho_n$. The continuity of f and (4.2.2) imply that there are $K' \Subset \mathbb{R}^d$, $h > 0$, and $C' > 0$ such that for all $K \Subset \mathbb{R}^d$ it holds that

$$\begin{aligned} |\langle f - \varphi_n, \psi \rangle| &\leq C' \|\psi * \rho_n - \psi\|_{\mathcal{E}_{M_p, h}(K')} \\ &\leq C' \sup_{\alpha \in \mathbb{N}^d} \max_{x \in K'} \frac{|(\partial^\alpha \psi) * \rho_n(x) - \partial^\alpha \psi(x)|}{h^{|\alpha|} M_\alpha} \\ &\leq CC' e^{-M(n/(\sqrt{d}hH))} \sup_{\alpha \in \mathbb{N}^d} \frac{\|\partial^\alpha \psi\|_{\mathcal{E}_{M_p, h}(K)}}{h^{|\alpha|} M_\alpha} \\ &\leq C_0 CC' \|\psi\|_{\mathcal{E}_{M_p, h/H}(K)} e^{-M(n/(\sqrt{d}hH))}, \end{aligned}$$

for all $\psi \in \mathcal{D}_K^{(M_p)}$. □

4.3 Regular generalized functions of class (M_p) and $\{M_p\}$

In this section, we define a notion of regularity with respect to ultradifferentiability of class $*$ in the algebra of generalized functions of class $*$ and show that an ultradistribution of class $*$ is a regular generalized function if and only if it is an ultradifferentiable function of class $*$.

Let $\Omega \subseteq \mathbb{R}^d$ be open. We define the algebra of *regular generalized functions of class $*$ in Ω* as

$$\mathcal{G}^{*,\infty}(\Omega) := \mathcal{E}_{\mathcal{M}}^{*,\infty}(\Omega) / \mathcal{E}_{\mathcal{N}}^*(\Omega),$$

where

$$\begin{aligned} \mathcal{E}_{\mathcal{M}}^{(M_p),\infty}(\Omega) := \{(\varphi_n)_n \in \mathcal{E}^{(M_p)}(\Omega)^{\mathbb{N}} \mid \forall K \Subset \Omega \exists \lambda > 0 \forall h > 0 : \\ \sup_{n \in \mathbb{N}} \|\varphi_n\|_{\mathcal{E}^{M_p,h}(K)} e^{-M(\lambda n)} < \infty\} \end{aligned}$$

$$\begin{aligned} \mathcal{E}_{\mathcal{M}}^{\{M_p\},\infty}(\Omega) := \{(\varphi_n)_n \in \mathcal{E}^{\{M_p\}}(\Omega)^{\mathbb{N}} \mid \forall K \Subset \Omega \exists h > 0 \forall \lambda > 0 : \\ \sup_{n \in \mathbb{N}} \|\varphi_n\|_{\mathcal{E}^{M_p,h}(K)} e^{-M(\lambda n)} < \infty\}. \end{aligned}$$

Clearly, $\mathcal{G}^{*,\infty}$ is a subsheaf of \mathcal{G}^* . A generalized function $f \in \mathcal{G}^*(\Omega)$ is said to be regular of class $*$ on an open subset Ω' of Ω if its restriction to Ω' belongs to $\mathcal{G}^{*,\infty}(\Omega')$. The *singular support of class $*$* of $f \in \mathcal{G}^*(\Omega)$, denoted by $\text{singsupp}_{g,*} f$, is defined as the complement in Ω of the largest open set on which f is regular of class $*$. We need the following simple but useful consequence of Lemma 2.2.9.

Lemma 4.3.1. *Let $f \in \Gamma_c(\mathbb{R}^d, \mathcal{G}^*)$ and let $(\varphi_n)_n$ be a compactly supported representative of f . Then, $f \in \mathcal{G}^{*,\infty}(\mathbb{R}^d)$ if and only if*

$$\exists \lambda > 0 \forall h > 0 (\exists h > 0 \forall \lambda > 0) : \sup_{n \in \mathbb{N}} \sup_{\xi \in \mathbb{R}^d} |\widehat{\varphi}_n(\xi)| e^{M(\xi/h) - M(\lambda n)} < \infty.$$

Let $(\rho_n)_n$ be an (N_p) -mollifier sequence, where N_p is a weight sequence satisfying the conditions from Lemma 4.2.1, and consider the associated embedding ι from Theorem 4.2.4. The next regularity theorem gives a precise characterization of the embedded image of $\mathcal{E}^*(\Omega)$ under the embedding ι in terms of the algebra $\mathcal{G}^{*,\infty}(\Omega)$.

Theorem 4.3.2. *Let $\Omega \subseteq \mathbb{R}^d$ be open. Then, $\mathcal{G}^{*,\infty}(\Omega) \cap \iota(\mathcal{D}'^*(\Omega)) = \iota(\mathcal{E}^*(\Omega))$.*

Proof. The inclusion $\iota(\mathcal{E}^*(\Omega)) \subseteq \mathcal{G}^{*,\infty}(\Omega) \cap \iota(\mathcal{D}'^*(\Omega))$ is obvious. Conversely, let $f \in \mathcal{D}'^*(\Omega)$ be such that $\iota(f) \in \mathcal{G}^{*,\infty}(\Omega)$. We may assume that f is compactly supported. By the classical Paley-Wiener-Schwartz theorem and Lemma 2.2.7, it suffices to show that $\sup_{\xi \in \mathbb{R}^d} |\widehat{f}(\xi)| e^{M(\xi/h)} < \infty$ for every $h > 0$ (for some $h > 0$). Let $(\varphi_n)_n$ be a compactly supported representative of $\iota_c(f)$. Choose $K \Subset \Omega$ such that $\text{supp } f \subseteq K$ and $\text{supp } \varphi_n \subseteq K$ for all $n \in \mathbb{N}$. Pick $\kappa \in \mathcal{D}^{(M_p)}(\Omega)$ such that $\kappa \equiv 1$ on a neighbourhood of K . Hence,

$$\widehat{f}(\xi) = \langle f(x) - \varphi_n(x), \kappa(x) e^{-ix\xi} \rangle + \widehat{\varphi}_n(\xi), \quad \xi \in \mathbb{R}^d,$$

for all $n \in \mathbb{N}$. Lemma 4.2.5 implies that there are $h_1 > 0$ and $C_1 > 0$ such that

$$\begin{aligned} & |\langle f(x) - \varphi_n(x), \kappa(x) e^{-ix\xi} \rangle| \\ & \leq C_1 \|\kappa(x) e^{-ix\xi}\|_{\mathcal{E}^{M_p, 2h_1}(K_x)} e^{-M(n/(2\sqrt{d}H^2h_1))} \\ & \leq C_1 \|\kappa\|_{\mathcal{E}^{M_p, h_1}(K)} e^{M(\xi/h_1) - M(n/(2\sqrt{d}H^2h_1))}, \end{aligned}$$

for all $n \in \mathbb{N}$. Set $c = (2\sqrt{d}H^2)^{-1}$. By combining the above inequality with Lemma 4.3.1 we obtain that

$$\exists h_1, \lambda > 0 \forall h_2 > 0 \exists C > 0 (\exists h_1, h_2 > 0 \forall \lambda > 0 \exists C > 0) :$$

$$|\widehat{f}(\xi)| \leq C(e^{M(\xi/h_1) - M(cn/h_1)} + e^{M(\lambda n) - M(\xi/h_2)}) \quad (4.3.1)$$

for all $n \in \mathbb{N}$. In the remainder of the proof, we only treat the Beurling case; the Roumieu case is similar. Let $h > 0$ be arbitrary

and set $k = c \min\{1, h/h_1\}$. Fix $\xi \in \mathbb{R}^d$ with $|\xi| > k/H$. Set $h_2 = \min\{h, k/(2\lambda H)\}/H$ and let n be the smallest natural number such that $n \geq H|\xi|/k$. Inequality 4.3.1 now implies that $|\widehat{f}(\xi)| \leq C_0 C e^{M(\xi/h)}$. \square

Remark 4.3.3. Theorem 4.3.2 considerably improves some of the results from [135, 136] in which regularity properties of ultradistributions of class $*$ are studied via the rate of growth of sequences of regularizations. Contrary to the aforementioned works where only sufficient conditions were obtained, Theorem 4.3.2 provides a complete characterization of ultradifferentiability of class $*$.

4.4 Wave front sets

We now introduce a notion of microlocal regularity of class $*$ that is inspired by our definition of $\mathcal{G}^{*,\infty}$ -regularity. Our main result asserts that this notion is a compatible extension of the wave front set of class $*$ of an ultradistribution of class $*$.

Let $f \in \Gamma_c(\mathbb{R}^d, \mathcal{G}^*)$. The set $\Sigma_g^*(f) \subseteq \mathbb{R}^d \setminus \{0\}$ is defined as the complement in $\mathbb{R}^d \setminus \{0\}$ of the set of all points ξ_0 for which there are an open conic neighbourhood Γ and a compactly supported representative $(\varphi_n)_n$ of f such that

$$\exists \lambda > 0 \forall h > 0 (\exists h > 0 \forall \lambda > 0) : \sup_{n \in \mathbb{N}} \sup_{\xi \in \Gamma} |\widehat{\varphi}_n(\xi)| e^{M(\xi/h) - M(\lambda n)} < \infty.$$

This definition is independent of the chosen compactly supported representative of f . Furthermore, $\Sigma_g^*(f)$ is a closed cone and, by Lemma 4.3.1, $\Sigma_g^*(f) = \emptyset$ if and only if $f \in \mathcal{G}^{*,\infty}(\Omega)$.

Lemma 4.4.1. *Let $\Omega \subseteq \mathbb{R}^d$ be open. Then, $\Sigma_g^*(uf) \subseteq \Sigma_g^*(f)$ for all $f \in \Gamma_c(\Omega, \mathcal{G}^*)$ and $u \in \mathcal{G}^{*,\infty}(\Omega)$.*

Proof. Let $(\varphi_n)_n$ be a compactly supported representative of f . Notice that $\|\widehat{\varphi}_n\|_{L^1} = O(e^{M(\lambda_0 n)})$ for some $\lambda_0 > 0$ (for every $\lambda_0 > 0$).

Next, choose $\kappa \in \mathcal{D}^*(\Omega)$ such that $\kappa \equiv 1$ in a neighbourhood of $\text{supp } f$, hence $uf = u\iota(\kappa)f$. Since $u\iota(\kappa) \in \mathcal{G}^{*,\infty}(\Omega) \cap \mathcal{G}_c^*(\Omega)$, Lemma 4.3.1 yields that

$$\begin{aligned} & \exists \lambda_1 > 0 \forall h_1 > 0 (\exists h_1 > 0 \forall \lambda_1 > 0) : \\ & \sup_{n \in \mathbb{N}} \sup_{\xi \in \mathbb{R}^d} |\widehat{\psi}_n(\xi)| e^{M(\xi/h_1) - M(\lambda_1 n)} < \infty, \end{aligned}$$

where $(\psi_n)_n$ is a compactly supported representative of $u\iota(\kappa)$. Now suppose that $\xi_0 \notin \Sigma_g^*(f)$. Then, there is an open conic neighbourhood Γ such that

$$\begin{aligned} & \exists \lambda_2 > 0 \forall h_2 > 0 (\exists h_2 > 0 \forall \lambda_2 > 0) : \\ & \sup_{n \in \mathbb{N}} \sup_{\xi \in \Gamma} |\widehat{\varphi}_n(\xi)| e^{M(\xi/h_2) - M(\lambda_2 n)} < \infty. \end{aligned}$$

Choose an open conic neighbourhood Γ_1 of ξ_0 such that $\overline{\Gamma}_1 \subseteq \Gamma \cup \{0\}$. Let $0 < c < 1$ be smaller than the distance between $\partial\Gamma$ and the intersection of Γ_1 with the unit sphere. Observe that $\{\eta \in \mathbb{R}^d \mid \exists \xi \in \Gamma_1 : |\xi - \eta| \leq c|\xi|\} \subseteq \Gamma$ and that $|\xi - \eta| \leq c|\xi|$ implies that $|\eta| \geq (1 - c)|\xi|$ for all $\xi, \eta \in \mathbb{R}^d$. Hence, for all $\xi \in \Gamma_1$,

$$\begin{aligned} & |\mathcal{F}(\psi_n \varphi_n)(\xi)| \\ & \leq \frac{1}{(2\pi)^d} \left(\int_{|\eta| \leq c|\xi|} + \int_{|\eta| > c|\xi|} \right) |\widehat{\psi}_n(\eta)| |\widehat{\varphi}_n(\xi - \eta)| d\eta \\ & \leq \frac{\|\widehat{\psi}_n\|_{L^1}}{(2\pi)^d} \sup_{|\xi - \eta| \leq c|\xi|} |\widehat{\varphi}_n(\eta)| + C \|\widehat{\varphi}_n\|_{L^1} e^{-M(c\xi/h_1) + M(\lambda_1 n)} \\ & \leq C' (e^{-M((1-c)\xi/h_2) + M(\lambda_1 n) + M(\lambda_2 n)} + e^{-M(c\xi/h_1) + M(\lambda_0 n) + M(\lambda_1 n)}) \\ & \leq C_0 C' (e^{-M((1-c)\xi/h_2) + M(H \max\{\lambda_1, \lambda_2\}n)} \\ & \quad + e^{-M(c\xi/h_1) + M(H \max\{\lambda_0, \lambda_1\}n)}), \end{aligned}$$

for all $n \in \mathbb{N}$ and some $C, C' > 0$. □

Let $\Omega \subseteq \mathbb{R}^d$ be open and let $f \in \mathcal{G}^*(\Omega)$. The generalized wave front set $WF_{g,*}(f)$ is defined as the complement in $\Omega \times (\mathbb{R}^d \setminus \{0\})$ of all pairs (x_0, ξ_0) for which there is $\psi \in \mathcal{D}^*(\Omega)$, with $\psi \equiv 1$ in a

neighbourhood of x_0 , such that $\xi_0 \notin \Sigma_g^*(\iota(\psi)f)$. In the Roumieu case, one may always take $\psi \in \mathcal{D}^{(M_p)}(\Omega)$, as follows from Lemma 4.4.1. Moreover, as in [79, Sect. 8.1], Lemma 4.4.1 also implies that the projection of $WF_{g,*}(f)$ on Ω is $\text{singsupp}_{g,*}(f)$ while its projection on $\mathbb{R}^d \setminus \{0\}$ is $\Sigma_g^*(f)$ if $f \in \Gamma_c(\mathbb{R}^d, \mathcal{G}^*)$.

We now compare $WF_{g,*}$ with its classical counterpart for ultra-distributions of class $*$. We follow the standard definition for the wave front set $WF_*(f)$ of $f \in \mathcal{D}'^*(\Omega)$ [93, 133]. Namely, the set $WF_*(f)$ is defined as the complement in $\Omega \times (\mathbb{R}^d \setminus \{0\})$ of the set of all pairs (x_0, ξ_0) for which there are an open conic neighbourhood Γ of ξ_0 and $\psi \in \mathcal{D}^*(\Omega)$, with $\psi \equiv 1$ in a neighbourhood of x_0 , such that

$$\sup_{\xi \in \Gamma} |\widehat{\psi f}(\xi)| e^{M(\xi/h)} < \infty$$

for all $h > 0$ (for some $h > 0$). Let $(\rho_n)_n$ be an (N_p) -mollifier sequence, where N_p is a weight sequence satisfying the conditions from Lemma 4.2.1, and consider the associated embedding ι from Theorem 4.2.4. We have the the following important equality; it is a microlocal refinement of Theorem 4.3.2.

Theorem 4.4.2. *Let $\Omega \subseteq \mathbb{R}^d$ be open and $f \in \mathcal{D}'^*(\Omega)$. Then, $WF_*(f) = WF_{g,*}(\iota(f))$.*

Proof. Let $(x_0, \xi_0) \notin WF_*(f)$. Find an open conic neighbourhood Γ of ξ_0 and $\psi \in \mathcal{D}^*(\Omega)$, with $\psi \equiv 1$ in a neighbourhood of x_0 , such that $\sup_{\xi \in \Gamma} |\widehat{\psi f}(\xi)| e^{M(\xi/h)} < \infty$ for every $h > 0$ (for some $h > 0$). Choose $\chi \in \mathcal{D}^*(\Omega)$ such that $\chi \equiv 1$ in a neighbourhood of x_0 and $\psi \equiv 1$ in a neighbourhood of $\text{supp } \chi$. We claim that $\xi_0 \notin \Sigma_g^*(\iota(\chi)\iota(f))$. Theorem 4.2.4 gives us that $\iota(\chi)\iota(f) = \iota(\chi)\iota(\psi f) = [(\chi(\psi f * \rho_n))_n]$. Lemma 2.2.9 yields that $\sup_{\xi \in \mathbb{R}^d} |\widehat{\chi}(\xi)| e^{M(\xi/h_1)} < \infty$ for every $h_1 > 0$ (for some $h_1 > 0$) while $\sup_{\xi \in \mathbb{R}^d} |\widehat{\psi f}(\xi)| e^{-M(\xi/h_2)} < \infty$ for some $h_2 > 0$ (for all $h_2 > 0$). Pick an open conic neighbour-

hood Γ_1 of ξ_0 as in Lemma 4.4.1. Hence, for all $\xi \in \Gamma_1$,

$$\begin{aligned}
& |\mathcal{F}(\chi(\psi f * \rho_n))(\xi)| \\
& \leq \frac{1}{(2\pi)^d} \left(\int_{|\eta| \leq c|\xi|} + \int_{|\eta| > c|\xi|} \right) |\widehat{\chi}(\eta)| |\widehat{\psi f}(\xi - \eta)| |\widehat{\rho}((\xi - \eta)/n)| d\eta \\
& \leq \frac{\|\widehat{\rho}\|_{L^\infty} \|\widehat{\chi}\|_{L^1}}{(2\pi)^d} \sup_{|\xi - \eta| \leq c|\xi|} |\widehat{\psi f}(\eta)| \\
& + \frac{C}{(2\pi)^d} e^{-M(c\xi/h_1)} \int_{\mathbb{R}^d} e^{M(\xi/h_2)} |\widehat{\rho}(\xi/n)| d\xi \\
& \leq \frac{C' \|\widehat{\rho}\|_{L^\infty} \|\widehat{\chi}\|_{L^1}}{(2\pi)^d} e^{-M((1-c)\xi/h)} + \frac{C_0 C \|\widehat{\rho}\|_{L^1}}{(2\pi)^d} e^{-M(c\xi/h_1) + M(2Hn/h_2)},
\end{aligned}$$

for $n \in \mathbb{N}$ large enough and some $C, C' > 0$, whence $(x_0, \xi_0) \notin WF_{g,*}(\iota(f))$. Conversely, let $(x_0, \xi_0) \notin WF_{g,*}(\iota(f))$. Find $\psi \in \mathcal{D}^{(M_p)}(\Omega)$, with $\psi \equiv 1$ in a neighbourhood of x_0 , such that $\xi_0 \notin \Sigma_g^*(\iota(\psi)\iota(f))$. Let $\kappa \in \mathcal{D}^*(\Omega)$ be such that $\kappa \equiv 1$ in a neighbourhood of $\text{supp } \psi$. We obtain from Theorem 4.2.4 that $\iota(\psi)\iota(f) = \iota(\psi)\iota(\kappa f) = [(\psi\chi_n)_n]$, where $(\chi_n)_n$ is a compactly supported representative of $\iota(\kappa f)$. Hence, there is an open conic neighbourhood Γ of ξ_0 such that

$$\begin{aligned}
& \exists \lambda > 0 \forall h > 0 (\exists h > 0 \forall \lambda > 0) : \\
& \sup_{n \in \mathbb{N}} \sup_{\xi \in \Gamma} |\widehat{\psi\chi_n}(\xi)| e^{M(\xi/h) - M(\lambda n)} < \infty.
\end{aligned} \tag{4.4.1}$$

We have that

$$\widehat{\psi f}(\xi) = \widehat{\psi \kappa f}(\xi) = \langle \kappa f(x) - \chi_n(x), \psi(x) e^{-ix\xi} \rangle + \widehat{\psi\chi_n}(\xi)$$

for all $n \in \mathbb{N}$. Exactly as in the proof of Theorem 4.3.2, but by using inequality (4.4) instead of Lemma 4.3.1, one can show that $\sup_{\xi \in \Gamma} |\widehat{\psi f}(\xi)| e^{M(\xi/h)} < \infty$ for all $h > 0$ (for some $h > 0$). The details are left to the reader. \square

Chapter 5

Solution to the first Cousin problem for vector-valued quasianalytic functions

5.1 Introduction

In this chapter, we present the solution to the first Cousin problem for vector-valued quasianalytic functions. The vector-valued results will be applied in the next chapter to show that, also in the quasianalytic case, the presheaf $\mathcal{G}^{\{M_p\}}$ is in fact a sheaf. Actually, this is our main motivation to study the Cousin problem but it is certainly also interesting in its own right.

In abstract terms, the first Cousin problem can be formulated as follows. Let X be a topological space and let \mathcal{F} be a sheaf on X . Let $U \subseteq X$ be open and let $(U_i)_i$ be an open covering of U . Suppose that $\varphi_{i,j} \in \Gamma(U_i \cap U_j, \mathcal{F})$ are given sections such that

$$\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} = 0 \quad \text{on } U_i \cap U_j \cap U_k$$

for all i, j, k . Are there $\varphi_i \in \Gamma(U_i, \mathcal{F})$ such that

$$\varphi_{i,j} = \varphi_j - \varphi_i \quad \text{on } U_i \cap U_j$$

for all i, j ? For $X = \mathbb{C}^d$ and \mathcal{F} the sheaf of holomorphic functions the Cousin problem is solvable if U is a Stein open set, as follows from the celebrated Oka-Cartan theorem. This problem was very important for the development of the modern theory of functions of several complex variables and led to the use of sheaf cohomology theory in that area. We refer to [74] for a clear exposition of the problem. Since every open set in \mathbb{R}^d has a system of complex neighbourhoods consisting of Stein open sets, it follows that the Cousin problem is solvable for $X = \mathbb{R}^d$ and \mathcal{F} the sheaf of real analytic functions (where U and U_i are now arbitrary open sets in \mathbb{R}^d). Petzsche announced in [129] the solution to the Cousin problem for quasianalytic functions in connection with the construction of sheaves of infrahyperfuctions, but his article on the subject seems not to have appeared. Our aim is to show that the Cousin problem is indeed solvable in spaces of quasianalytic functions. We shall also give sufficient conditions on a locally convex space F such that the Cousin problem is solvable in spaces of F -valued quasianalytic functions.

The analysis of the Cousin problem requires the study of topological properties of spaces of quasianalytic functions. The space of real analytic functions has been thoroughly investigated in the literature and its locally convex structure is by now well understood; see [110, 17, 58] for the scalar-valued case and [100, 15, 16] for the vector-valued case. Much less is known for general spaces of quasianalytic functions, although some work has been done [140, 17, 102]. The first part of this chapter is devoted to the study of various useful topological properties of spaces of vector-valued quasianalytic functions. Even in the scalar-valued case, some of the results we discuss here appear to be new; for example, we will show that the spaces of quasianalytic functions of Roumieu type are ultrabornological (PLN)-spaces, a fact that, to the best of our knowledge, remained unnoticed in the literature for general open subsets of \mathbb{R}^d .

Finally, we would like to express our gratitude to the authors of [17, 55], as many of our proofs below rely on their results. In particular, Domański's impressive work [55] on the ε -product of (PLS) -spaces was very inspiring to us.

This chapter is organized as follows. In section 5.2, we discuss the locally convex structure of spaces of quasianalytic functions of Roumieu type. In particular, we show that these spaces are ultrabornological and give an explicit system of seminorms generating their topology; such a projective description plays an important role in the analysis of the vector-valued case (cf. [92]). Spaces of vector-valued quasianalytic functions are studied in Section 5.3, we closely follow Komatsu's approach from [92]. Finally, the Cousin problem is discussed in Section 5.4.

Unless explicitly stated otherwise, M_p will in this chapter stand for a weight sequence satisfying $(M.1)$, $(M.2)'$, (QA) , and (NA) in the Beurling case and $(M.1)$, $(M.2)'$, (QA) , and (NE) in the Roumieu case.

5.2 Topological properties of spaces of quasianalytic functions of Roumieu type

In this section, we discuss some of the topological properties of the spaces of quasianalytic functions of Roumieu type. Firstly, we show that $\mathcal{E}^{\{M_p\}}(\Omega)$ is an ultrabornological (PLN) -space for any open subset Ω of \mathbb{R}^d . Our proof is based on Theorem 2.3.7. Next, we provide a projective description of $\mathcal{E}^{\{M_p\}}(\Omega)$, thereby extending a classical result of Komatsu [92, Prop. 3.5] to the quasianalytic case. For this, we shall generalize Komatsu's first structural theorem [89, Thm. 8.1] to quasianalytic functionals.

Proposition 5.2.1. *Let $\Omega \subseteq \mathbb{R}^d$ be open. Then, the space $\mathcal{E}^{\{M_p\}}(\Omega)$ is an ultrabornological (PLN)-space.*

Proof. Let $(K_N)_N$ be an exhaustion by compact sets of Ω . The projective spectrum $\mathcal{X} = (\mathcal{E}^{\{M_p\}}[K_N])_N$ (with canonical linking mappings) consists of (DFN)-spaces [89, Prop. 2.5] and is reduced. Moreover, we have the following isomorphism of l.c.s.

$$\mathcal{E}^{\{M_p\}}(\Omega) \cong \varprojlim_{N \in \mathbb{N}} \mathcal{E}^{\{M_p\}}[K_N],$$

whence $\mathcal{E}^{\{M_p\}}(\Omega)$ is a (PLN)-space. We now show that it is ultrabornological. By Theorem 2.3.7 it suffices to show that the dual spectrum \mathcal{X}^* is α -regular. Let $B \subset \mathcal{E}'^{\{M_p\}}(\Omega)$ be bounded. It is a classical result of Martineau that $\mathcal{A}(\Omega)$ is ultrabornological [110, Thm. 1.2 and Prop. 1.9]. Since the inclusion mapping $\mathcal{E}'^{\{M_p\}}(\Omega) \rightarrow \mathcal{A}'(\Omega)$ is continuous, B is bounded in $\mathcal{A}'(\Omega)$ and, thus, $B \subset \mathcal{A}'[K_N]$ for some $N \in \mathbb{N}$. By Proposition 2.2.10 we may conclude that $B \subset \mathcal{E}'^{\{M_p\}}[K_N]$. \square

Remark 5.2.2. In the non-quasianalytic case, Proposition 5.2.1 is due to Komatsu [89, Thm. 5.12]. In the real analytic case, Proposition 5.2.1 was shown by Martineau [110, Thm. 1.2 and Prop. 1.9]; in fact, we used this result in the proof of Proposition 5.2.1. Finally, for Ω convex, Proposition 5.2.1 is due to Rösner [140] who used a Paley-Wiener type result for quasianalytic functionals to transform it into a problem concerning weighted (LF)-spaces of entire functions.

Next, we turn our attention to the projective description of the space $\mathcal{E}^{\{M_p\}}(\Omega)$. Our goal is to show the following result.

Theorem 5.2.3. *Let $\Omega \subseteq \mathbb{R}^d$ be open. Then, a function $\varphi \in C^\infty(\Omega)$ belongs to $\mathcal{E}^{\{M_p\}}(\Omega)$ if and only if*

$$\|\varphi\|_{\mathcal{E}^{M_p, h_j}(K)} := \sup_{\alpha \in \mathbb{N}^d} \max_{x \in K} \frac{|\partial^\alpha \varphi(x)|}{M_\alpha \prod_{j=0}^{|\alpha|} h_j} < \infty$$

for all $K \Subset \Omega$ and $h_j \in \mathfrak{R}$. Moreover, the topology of $\mathcal{E}^{\{M_p\}}(\Omega)$ is generated by the system of seminorms $\{\|\cdot\|_{\mathcal{E}^{M_p, h_j}(K)} \mid K \Subset \Omega, h_j \in \mathfrak{R}\}$.

Following Komatsu [92, Prop. 3.5], our proof of Theorem 5.2.3 is based on the ensuing structural theorem for $\mathcal{E}^{\{M_p\}}(\Omega)$.

Proposition 5.2.4. *Let $\Omega \subseteq \mathbb{R}^d$ be open. For every bounded set $B \subset \mathcal{E}^{\{M_p\}}(\Omega)$ there are $K \Subset \Omega$ and regular complex Borel measures $\mu_\alpha(f) \in C'(K)$, $\alpha \in \mathbb{N}^d$, $f \in B$, such that*

$$\sup_{f \in B} \sup_{\alpha \in \mathbb{N}^d} \|\mu_\alpha(f)\|_{C'(K)} M_\alpha \prod_{j=0}^{|\alpha|} h_j < \infty$$

for some $h_j \in \mathfrak{R}$ and

$$f = \sum_{\alpha \in \mathbb{N}^d} \partial^\alpha \mu_\alpha(f), \quad f \in B.$$

Before we prove Proposition 5.2.4, let us show how Theorem 5.2.3 follows from it. In fact, the proof becomes identical to that of [92, Prop. 3.5]. We repeat the argument for the sake of completeness.

Proof of Theorem 5.2.3. The first statement follows from Lemma 2.2.2(i), we thus only have to check the topological assertion. Every seminorm $\|\cdot\|_{\mathcal{E}^{M_p, h_j}(K)}$ acts continuously on $\mathcal{E}^{\{M_p\}}(\Omega)$. Conversely, let $q \in \text{csn}(\mathcal{E}^{\{M_p\}}(\Omega))$ be arbitrary. There is a bounded set $B \subset \mathcal{E}^{\{M_p\}}(\Omega)$ such that $q(\varphi) \leq \sup_{f \in B} |\langle f, \varphi \rangle|$ for all $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega)$. Choose $K \Subset \Omega$ and $\mu_\alpha(f) \in C'(K)$, $\alpha \in \mathbb{N}^d$, $f \in B$, as in Lemma 5.2.4. Hence, there is $C > 0$ such that

$$\begin{aligned} q(\varphi) &\leq \sup_{f \in B} \sum_{\alpha \in \mathbb{N}^d} \|\mu_\alpha(f)\|_{C'(K)} \|\partial^\alpha \varphi\|_{C(K)} \\ &\leq C \sum_{\alpha \in \mathbb{N}^d} \frac{\|\partial^\alpha \varphi\|_{C(K)}}{M_\alpha \prod_{j=0}^{|\alpha|} h_j} \leq 2^d C \|\varphi\|_{\mathcal{E}^{M_p, h'_j}(K)}, \end{aligned}$$

where $h'_j = h_j/2$, for all $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega)$. □

The rest of this section is devoted to the proof of Proposition 5.2.4. Our first observation is that Komatsu's proof of the first structural theorem for ultradistributions of Beurling type [89, Thm. 8.1] does not depend on the assumption $(M.3)'$. More precisely, we have that:

Proposition 5.2.5. *Let M_p be a weight sequence satisfying (M.1) and let $\Omega \subseteq \mathbb{R}^d$ be open. Then, for every bounded set $B \subset \mathcal{E}'^{(M_p)}(\Omega)$ there are $K \Subset \Omega$ and regular complex Borel measures $\mu_\alpha(f) \in C'(K)$, $\alpha \in \mathbb{N}^d$, $f \in B$, such that*

$$\sup_{f \in B} \sup_{\alpha \in \mathbb{N}^d} \|\mu_\alpha(f)\|_{C'(K)} M_\alpha h^{|\alpha|} < \infty$$

for some $h > 0$ and

$$f = \sum_{\alpha \in \mathbb{N}^d} \partial^\alpha \mu_\alpha(f), \quad f \in B.$$

Our strategy is to reduce Proposition 5.2.4 to Proposition 5.2.5 with the aid of a Paley-Wiener type result for quasianalytic functionals and Hörmander's support splitting theorem (Proposition 2.2.14). We need some preparation. The *support function* [117, Sect. 1.8] of a convex compact set K in \mathbb{R}^d is defined as

$$h_K(\xi) := \max_{x \in K} \operatorname{Re}(x\xi), \quad \xi \in \mathbb{C}^d.$$

For $h > 0$ we write $\mathcal{O}_K^{M_p, h}$ for the Banach space of all entire functions $F \in \mathcal{O}(\mathbb{C}^d)$ such that

$$\sup_{\xi \in \mathbb{C}^d} |F(\xi)| e^{-h_K(\xi) - M(\xi/h)} < \infty.$$

For a convex open set Ω in \mathbb{R}^d we define

$$\mathcal{O}_\Omega^{(M_p)} := \varinjlim_{K \Subset \Omega} \varinjlim_{h \rightarrow 0^+} \mathcal{O}_K^{M_p, h}, \quad \mathcal{O}_\Omega^{\{M_p\}} := \varinjlim_{K \Subset \Omega} \varprojlim_{h \rightarrow \infty} \mathcal{O}_K^{M_p, h}.$$

We then have the following result.

Proposition 5.2.6. ([154, Thm. 4.1]) *Let Ω be a convex open set in \mathbb{R}^d . Then, the Fourier transform $\mathcal{F} : \mathcal{E}'^*(\Omega) \rightarrow \mathcal{O}_\Omega^*$ is a topological isomorphism.*

Lemma 5.2.7. *Let $h_j \in \mathfrak{R}$ and let M_p be a weight sequence satisfying (M.1), (M.2)', and (QA). Then, there is $h'_j \in \mathfrak{R}$ with $h'_j \preceq h_j$ such that the sequence $M_p \prod_{j=0}^p h'_j$ also satisfies (M.1), (M.2)', and (QA).*

Proof. Set

$$k_j = 1 + \left(\sum_{p=1}^j \frac{1}{m_p} \right)^{1/2}, \quad j \in \mathbb{N}.$$

The sequence $h'_j \in \mathfrak{R}$ given by $h'_j = \min\{h_j, 2^j, k_j\}$, $j \in \mathbb{N}$, satisfies all requirements. \square

Proof of Proposition 5.2.4. In view of Proposition 5.2.5, it suffices to show that for every bounded set B in $\mathcal{E}'^{\{M_p\}}(\Omega)$ there is $h_j \in \mathfrak{R}$ such that B is contained and bounded in $\mathcal{E}'^{(N_p)}(\Omega)$, where $N_p = M_p \prod_{j=0}^p h_j$. We shall prove this in two steps.

STEP I: Ω is convex. Propositions 5.2.1 and 5.2.6 imply that there is a convex compact set K in Ω such that

$$\sup_{f \in B} \sup_{\xi \in \mathbb{C}^d} |\widehat{f}(\xi)| e^{-h_K(\xi) - M(\xi/h)} < \infty$$

for all $h > 0$. By Lemma 2.2.4 we find a sequence $h_j \in \mathfrak{R}$ such that

$$\sup_{f \in B} \sup_{\xi \in \mathbb{C}^d} |\widehat{f}(\xi)| e^{-h_K(\xi) - M_{h_j}(\xi)} < \infty.$$

We may assume that $M_p \prod_{j=0}^p h_j$ satisfies (M.1), (M.2)', and (QA) by Lemma 5.2.7. Another application of Proposition 5.2.6 now yields the desired result.

STEP II: Ω is arbitrary. By Proposition 5.2.1 there is $K \Subset \Omega$ such that B is contained and bounded in $\mathcal{E}'^{\{M_p\}}[K]$. Let K_1, \dots, K_n , $n \in \mathbb{N}$, be convex compact sets in Ω such that $K \subseteq \bigcup_{j=1}^n K_j$. Proposition 2.2.14 implies that there are bounded sets $B_j \subset \mathcal{E}'^{\{M_p\}}[K_j]$,

$j = 1, \dots, n$, such that $B = B_1 + \dots + B_n$. The result now follows from the first step. \square

Remark 5.2.8. The technique of reducing the case of arbitrary open sets to convex open sets employed in the above proof is due to Heinrich and Meise [76].

5.3 Vector-valued quasianalytic functions

We now turn our attention to spaces of vector-valued quasianalytic functions. Our first goal is to derive an ε -product representation of these spaces. Based upon this representation, we shall discuss the topological properties of spaces of vector-valued quasianalytic of Roumieu type by making use of a deep result of Domański on the ε -product of (PLS) -spaces [55] and an (Ω) -type topological invariant of the space of real analytic functions [17]. We start by collecting various results about the ε -product of (PLS) -spaces.

5.3.1 The ε -product of (PLS) -spaces

Let X and Y be locally convex spaces. We write $L_c(X, Y)$ to indicate that we endow $L(X, Y)$ with the topology of uniform convergence on the balanced convex compact sets of X . The space $L(X'_c, Y)$ endowed with the topology of uniform convergence on the equicontinuous subsets of X' is denoted by $L_\varepsilon(X'_c, Y)$.

Following Schwartz [150] and Komatsu [92], we denote by $X\varepsilon Y$ the space consisting of all bilinear functionals on $X'_c \times Y'_c$ which are hypocontinuous with respect to the equicontinuous subsets of X' and Y' . We endow $X\varepsilon Y$ with the topology of uniform convergence on the products of equicontinuous subsets of X' and Y' . As pointed out in [92, p. 657], the following isomorphisms of l.c.s. hold

$$X\varepsilon Y \cong L_\varepsilon(X'_c, Y) \cong L_\varepsilon(Y'_c, X).$$

The tensor product $X \otimes Y$ is embedded in $X \varepsilon Y$ via $(x \otimes y)(x', y') = \langle x', x \rangle \langle y', y \rangle$. Clearly, the induced topology on $X \otimes Y$ is the ε -topology. If X and Y are complete and if either X or Y has the weak approximation property (in particular, if either X or Y is nuclear) we have that $X \varepsilon Y = X \widehat{\otimes}_\varepsilon Y$ [92, Prop. 1.4]. As usual, we write $X \widehat{\otimes} Y := X \widehat{\otimes}_\varepsilon Y = X \widehat{\otimes}_\pi Y$ if either X or Y is nuclear.

Let X_i and Y_i , $i = 1, 2$, be locally convex spaces and let $T_1 \in L(X_1, Y_1)$ and $T_2 \in L(X_2, Y_2)$. We write $T_1 \varepsilon T_2 : X_1 \varepsilon X_2 \rightarrow Y_1 \varepsilon Y_2$ for the continuous linear mapping given by $T_1 \varepsilon T_2(\Phi)(y'_1, y'_2) := \Phi({}^t T_1 y'_1, {}^t T_2 y'_2)$. The restriction of $T_1 \varepsilon T_2$ to $X_1 \otimes X_2$ is equal to the tensor product of the mappings T_1 and T_2 .

Next, we discuss the ε -product of (PLS) -spaces. Let $X = \varprojlim_N X_N$ be a (PLN) -space with $(X_N)_N$ a reduced projective spectrum of (DFN) -spaces and let $Y = \varprojlim_N Y_N$ be a (PLS) -space with $(Y_N)_N$ a reduced projective spectrum of (DFS) -spaces. By [92, Prop. 1.5] we have the following isomorphism of l.c.s.

$$X \varepsilon Y \cong \varprojlim_{N \in \mathbb{N}} X_N \varepsilon Y_N.$$

The ε -product of two (DFS) -spaces is again a (DFS) -space [12, Prop. 4.3] and $X \varepsilon Y = X \widehat{\otimes} Y$ is dense in each $X_N \varepsilon Y_N = X_N \widehat{\otimes} Y_N$. Hence, $X \varepsilon Y$ is a (PLS) -space that can be represented as the projective limit of the reduced spectrum $(X_N \varepsilon Y_N)_N$ of (DFS) -spaces.

The following result shall be used to deduce the vector-valued version of the Cousin problem from the solution of the scalar-valued one.

Lemma 5.3.1. ([55, Prop. 4.5]) *Let*

$$0 \longrightarrow X \xrightarrow{S} Y \xrightarrow{T} Z \longrightarrow 0$$

be a topologically exact sequence of (PLS) -spaces and let F be a (PLS) -space. Suppose that X is a (PLN) -space and that $X \varepsilon F$ is ultrabornological. Then, the sequence

$$0 \longrightarrow X_{\varepsilon}F \xrightarrow{S\varepsilon \text{id}_F} Y_{\varepsilon}F \xrightarrow{T\varepsilon \text{id}_F} Z_{\varepsilon}F \longrightarrow 0$$

is exact.

For Lemma 5.3.1 to be applicable in concrete situations, it is necessary to find verifiable conditions on X and F that ensure that $X_{\varepsilon}F$ is ultrabornological. Domański achieved this goal in [55]. He formulated such conditions in terms of the dual interpolation estimates for (PLS) -spaces [17] and used them to study the problem of real analytic parameter dependence of solutions of linear partial differential equations. We now discuss the precise definitions. Let $X = \varprojlim X_N$ be a (PLS) -space with $(X_N)_N$ a reduced projective spectrum of (DFS) -spaces. Suppose that $X_N = \varinjlim_n X_{N,n}$ and denote by $\|\cdot\|_{N,n}$ the norm of $X_{N,n}$. The dual norm of $\|\cdot\|_{N,n}$ is given by

$$\|x'\|_{N,n}^* := \sup\{\langle x', x \rangle \mid x \in X_{N,n}, \|x\|_{N,n} \leq 1\}, \quad x' \in X'_{N,n}.$$

The space X is said to satisfy the *dual interpolation estimate for small theta* if

$$\begin{aligned} \forall N \exists M \geq N \forall K \geq M \exists n \forall m \geq n \exists \theta_0 \in (0, 1) \\ \forall \theta \in (0, \theta_0) \exists k \geq m \exists C > 0 \forall x' \in X'_N : \\ \|x'\|_{M,m}^* \leq C (\|x'\|_{K,k}^*)^{1-\theta} (\|x'\|_{N,n}^*)^{\theta}. \end{aligned}$$

If a (PLS) -space satisfies the dual interpolation estimate for small theta, then its dual has (wQ) (cf. Subsection 2.3.1). Hence, Theorem 2.3.7 yields that every (PLS) -space satisfying the dual interpolation estimate for small theta is ultrabornological. Domański showed the following deep and very useful result; it may be seen as a generalization of the celebrated (DN) - (Ω) splitting theorem of Vogt and Wagner [162, Satz 1.9], [158, Thm. 5.1].

Theorem 5.3.2. ([55, Thm. 5.6]) *Let X be a (PLN) -space and let Y a (PLS) -space. Suppose that both X and Y satisfy the dual interpolation estimate for small theta. Then, $X\varepsilon Y$ is ultrabornological.*

Remark 5.3.3. Domański showed the above result under the additional assumption that the space X is deeply reduced (we refer to [55, p. 194] for the definition of this notion). By [137, Prop. 8] this assumption is superfluous. In the same paper, Theorem 5.3.2 was improved in the following way: Let X and Y be (PLS) -spaces satisfying the conditions from Theorem 5.3.2. Then, $X\varepsilon Y$ satisfies the dual interpolation estimate for small theta [137, Thm. 9].

Finally, we reformulate the dual interpolation estimate for small theta for (DFS) -spaces. A Fréchet space X with a fundamental increasing sequence of seminorms $(\|\cdot\|_n)_n$ is said to satisfy the condition (\underline{DN}) [114, p. 368] if

$$\exists n \forall m \geq n \exists \theta \in (0, 1) \exists k \geq m \exists C > 0 \forall x \in X : \\ \|x\|_m \leq C \|x\|_k^{1-\theta} \|x\|_n^\theta.$$

Hence, a (DFS) -space satisfies the dual interpolation estimate for small theta if and only if its strong dual satisfies (\underline{DN}) .

5.3.2 ε -product representations

Let F be a l.c.s. and let $\Omega \subseteq \mathbb{R}^d$ be open. We write $\mathcal{E}^{(M_p)}(\Omega; F)$ for the space consisting of all $\varphi \in C^\infty(\Omega; F)$ such that for all $q \in \text{csn}(F)$, $K \Subset \Omega$, and $h > 0$ it holds that

$$q_{M_p, h, K}(\varphi) := \sup_{\alpha \in \mathbb{N}^d} \max_{x \in K} \frac{q(\partial^\alpha \varphi(x))}{h^{|\alpha|} M_\alpha} < \infty.$$

Similarly, we define $\mathcal{E}^{\{M_p\}}(\Omega; F)$ as the space consisting of all $\varphi \in C^\infty(\Omega; F)$ such that for all $q \in \text{csn}(F)$, $K \Subset \Omega$, and $h_j \in \mathfrak{R}$ it holds that

$$q_{M_p, h_j, K}(\varphi) := \sup_{\alpha \in \mathbb{N}^d} \max_{x \in K} \frac{q(\partial^\alpha \varphi(x))}{M_\alpha \prod_{j=0}^{|\alpha|} h_j} < \infty.$$

By Lemma 2.2.2(i), the space $\mathcal{E}^{\{M_p\}}(\Omega; F)$ consists precisely of all $\varphi \in C^\infty(\Omega; F)$ such that for all $q \in \text{csn}(F)$ and $K \Subset \Omega$ there is $h > 0$ such that $q_{M_p, h, K}(\varphi) < \infty$. We endow the spaces $\mathcal{E}^{(M_p)}(\Omega; F)$ and $\mathcal{E}^{\{M_p\}}(\Omega; F)$ with the topology generated by the system of seminorms $\{q_{M_p, h, K} \mid q \in \text{csn}(F), K \Subset \Omega, h > 0\}$ and $\{q_{M_p, h_j, K} \mid q \in \text{csn}(F), K \Subset \Omega, h_j \in \mathfrak{R}\}$, respectively. The goal of this subsection is to show the ensuing result.

Proposition 5.3.4. *Let F be a sequentially complete l.c.s. and let $\Omega \subseteq \mathbb{R}^d$ be open. Then, $\mathcal{E}^*(\Omega; F)$ coincides with the space of all functions $\varphi : \Omega \rightarrow F$ such that $\langle y', \varphi(\cdot) \rangle \in \mathcal{E}^*(\Omega)$ for all $y' \in F'$. Moreover, we have the following canonical isomorphism of l.c.s.*

$$\mathcal{E}^*(\Omega; F) \cong \mathcal{E}^*(\Omega) \varepsilon F,$$

If, in addition, F is complete, then

$$\mathcal{E}^*(\Omega; F) \cong \mathcal{E}^*(\Omega) \varepsilon F \cong \mathcal{E}^*(\Omega) \widehat{\otimes} F.$$

Remark 5.3.5. In the non-quasianalytic case, Proposition 5.3.4 is due to Komatsu [92, Thm. 3.10]. Furthermore, in the real analytic case, Proposition 5.3.4 has been shown by Bonet and Domański [15, Thm. 16].

Following Komatsu, our proof of Proposition 5.3.4 will be based on the following general criterium.

Lemma 5.3.6. ([92, Lemma 1.12]) *Let E be a space consisting of complex-valued continuous functions on Ω endowed with a locally convex topology that is semi-Montel and stronger than the topology of uniform convergence on compact subsets of Ω and let F be a sequentially complete locally convex space. Suppose that the sequential closure¹ in E'_c of the set of functionals represented by measures with*

¹Let X be a topological space. The sequential closure of a subset $A \subseteq X$ is defined as the smallest sequentially closed set containing A . We remark that the sequential closure of A can be strictly larger than the sequential limit set of A (= set consisting of all $x \in X$ such that there is a sequence in A converging to x).

compact support in Ω is equal to E' . Then, $E \varepsilon F \cong L(F'_c, E)$ may be identified with the space consisting of all functions $\varphi : \Omega \rightarrow F$ such that $\langle y', \varphi(\cdot) \rangle \in E$ for all $y' \in F'$.

Lemma 5.3.7. *The sequential closure in $\mathcal{E}_b'^*(\Omega)$ of $\text{span}\{\delta_x \mid x \in \Omega\}$ is equal to $\mathcal{E}'^*(\Omega)$.*

Proof. Beurling case. The sequential closure of a set in a (DFS) -space is equal to its closure [19, Prop. 8.5.28]. Hence, it suffices to show that $\text{span}\{\delta_x \mid x \in \Omega\}$ is dense in $\mathcal{E}_b'^{(M_p)}(\Omega)$. By the Hahn-Banach theorem this follows from the reflexivity of $\mathcal{E}^{(M_p)}(\Omega)$.

Roumieu case. Let $(K_N)_N$ be an exhaustion by regular compact sets of Ω . We have that $\mathcal{E}_b'^{(M_p)}(\Omega) = \varinjlim_N \mathcal{E}_b'^{(M_p)}[K_N]$. As the spaces $\mathcal{E}_b'^{(M_p)}[K_N]$ are Fréchet spaces, it is enough to show that $\text{span}\{\delta_x \mid x \in K_N\}$ is dense in $\mathcal{E}_b'^{(M_p)}[K_N]$ for each $N \in \mathbb{N}$. The condition (QA) implies that an element $\varphi \in \mathcal{E}^{(M_p)}[K_N]$ is equal to zero if and only if one (and hence all) of its representatives vanishes on K_N . As in the Beurling case, the result now follows from the Hahn-Banach theorem and the reflexivity of $\mathcal{E}^{(M_p)}[K_N]$. \square

Proof of Proposition 5.3.4. In view of Lemma 5.3.7, the proof in fact becomes identical to that of [92, Thm. 3.10]. For the sake of completeness and to highlight the importance of the projective description of the space $\mathcal{E}^{\{M_p\}}(\Omega)$ (Theorem 5.2.3), we repeat the argument. We only show the Roumieu case; the Beurling case is similar. Clearly, $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega; F)$ implies that $\langle y', \varphi(\cdot) \rangle \in \mathcal{E}^{\{M_p\}}(\Omega)$ for all $y' \in F'$. Conversely, let $\varphi : \Omega \rightarrow F$ be a function having the latter property. In particular, it holds that $\langle y', \varphi(\cdot) \rangle \in C^\infty(\Omega)$ for all $y' \in F'$. A classical result of Grothendieck [148, App. Lemme II] implies that $\varphi \in C^\infty(\Omega; F)$ and that

$$\partial^\alpha \langle y', \varphi(\cdot) \rangle = \langle y', \partial^\alpha \varphi(\cdot) \rangle, \quad y' \in F', \quad (5.3.1)$$

for all $\alpha \in \mathbb{N}^d$. By Theorem 5.2.3 we obtain that, for all $K \Subset \Omega$ and $h_j \in \mathfrak{R}$, the set

$$\left\{ \frac{\partial^\alpha \varphi(x)}{M_\alpha \prod_{j=0}^{|\alpha|} h_j} \mid x \in K, \alpha \in \mathbb{N}^d \right\}$$

is weakly bounded and, thus, bounded in F . This means that $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega; F)$. Lemmas 5.3.6 and 5.3.7 therefore yield the algebraic isomorphism $\mathcal{E}^{\{M_p\}}(\Omega; F) \cong \mathcal{E}^{\{M_p\}}(\Omega) \varepsilon F$. Next, we show that this isomorphism also holds topologically. Let $q \in \text{csn}(F)$, $K \Subset \Omega$, and $h_j \in \mathfrak{R}$ be arbitrary. Define A to be the polar set of the $\|\cdot\|_{\mathcal{E}^{M_p, h_j}(K)}$ -unit ball in $\mathcal{E}^{\{M_p\}}(\Omega)$ and B to be the polar set of the q -unit ball in F . Hence, by (5.3.1) and the bipolar theorem, we obtain that

$$\begin{aligned} & \sup\{|\langle f, \langle y', \varphi(\cdot) \rangle \rangle| \mid f \in A, y' \in B\} \\ &= \sup\{\|\langle y', \varphi(\cdot) \rangle\|_{\mathcal{E}^{M_p, h_j}(K)} \mid y' \in B\} \\ &= \sup\left\{ \frac{|\langle y', \partial^\alpha \varphi(x) \rangle|}{M_\alpha \prod_{j=0}^{|\alpha|} h_j} \mid y' \in B, x \in K, \alpha \in \mathbb{N}^d \right\} \\ &= q_{M_p, h_j, K}(\varphi), \end{aligned}$$

whence the result follows from Theorem 5.2.3. The last part is a consequence of the completeness and nuclearity of $\mathcal{E}^{\{M_p\}}(\Omega)$ (cf. Subsection 5.3.1). \square

5.3.3 Topological properties of spaces of vector-valued quasianalytic functions of Roumieu type

Our next aim is to generalize Proposition 5.2.1 to the vector-valued case.

Proposition 5.3.8. *Let F be a (DFS) -space such that F' satisfies (\underline{DN}) and let $\Omega \subseteq \mathbb{R}^d$ be open. Then, $\mathcal{E}^{\{M_p\}}(\Omega; F)$ is an ultra-bornological (PLS) -space.*

We shall use a similar technique as in Proposition 5.2.1 to show Proposition 5.3.8. Therefore, we start with a discussion about the real analytic case. In this regard, the following result due to Bonet and Domański is very important.

Proposition 5.3.9. ([17, Cor. 2.2]) *Let $\Omega \subseteq \mathbb{R}^d$ be open. The space $\mathcal{A}(\Omega)$ satisfies the dual interpolation estimate for small theta.*

Theorem 5.3.2 and Proposition 5.3.4 imply the following corollary.

Corollary 5.3.10. *Let F be a (PLS) -space satisfying the dual interpolation estimate for small theta and let $\Omega \subseteq \mathbb{R}^d$ be open. Then, $\mathcal{A}(\Omega; F)$ is ultrabornological. In particular, this is the case if F is a (DFS) -space such that F' satisfies (DN) .*

Proof of Proposition 5.3.8. Let $(K_N)_N$ be an exhaustion by compact sets of Ω . The projective spectrum $\mathcal{X} = (\mathcal{E}^{\{M_p\}}[K_N] \widehat{\otimes} F)_N$ (with canonical linking mappings) consists of (DFS) -spaces and is reduced. By Proposition 5.3.4 and [92, Prop. 1.5], we have the following isomorphism of l.c.s.

$$\mathcal{E}^{\{M_p\}}(\Omega; F) \cong \varprojlim_{N \in \mathbb{N}} \mathcal{E}^{\{M_p\}}[K_N] \widehat{\otimes} F.$$

Hence, by Theorem 2.3.7, it suffices to show that the dual spectrum \mathcal{X}^* is α -regular. Let $B \subset (\mathcal{E}^{\{M_p\}}(\Omega; F))'$ be bounded. Since the inclusion mapping $(\mathcal{E}^{\{M_p\}}(\Omega; F))' \rightarrow (\mathcal{A}(\Omega; F))'$ is continuous, B is bounded in $(\mathcal{A}(\Omega; F))'$ and, thus, by Corollary 5.3.10, $B \subset (\mathcal{A}[K_N] \widehat{\otimes} F)'$ for some $N \in \mathbb{N}$. We claim that $B \subset (\mathcal{E}^{\{M_p\}}[K_N] \widehat{\otimes} F)'$. By [92, Prop. 2.3] it holds that

$$(\mathcal{E}^{\{M_p\}}[K_N] \widehat{\otimes} F)' \cong \mathcal{E}'^{\{M_p\}}[K_N] \widehat{\otimes} F' \cong L(F, \mathcal{E}'^{\{M_p\}}[K_N]).$$

In particular,

$$(\mathcal{E}^{\{M_p\}}(\Omega; F))' \cong \varprojlim_{N \in \mathbb{N}} L(F, \mathcal{E}'^{\{M_p\}}[K_N]) \subset L(F, \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)).$$

Hence, it suffices to observe that

$$L(F, \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)) \cap L(F, \mathcal{A}'[K_N]) \subseteq L(F, \mathcal{E}'^{\{M_p\}}[K_N]),$$

as follows from the Pták closed graph theorem and Proposition 2.2.10. \square

5.4 The Cousin problem

We are finally in the position to solve the Cousin problem. In the scalar-valued case, our proof is based on the vanishing of the derived projective limit functor for ultrabornological (*PLS*)-spaces (Theorem 2.3.7). The result is then extended to the vector-valued case by making use of Lemma 5.3.1 and the results from Section 5.3. It is natural to formulate the Cousin problem in the language of cohomology groups with coefficients in a sheaf. We therefore start with a brief discussion about the basic notions of this theory. For a detailed exposition we refer to [117, Chap. 4].

5.4.1 Cohomology groups with coefficients in a sheaf

Let X be a topological space and let \mathcal{F} be a sheaf on X . Let $\mathcal{M} = \{U_i \mid i \in I\}$ be a collection of open subsets of X . We write $U_{i_0, \dots, i_p} = U_{i_0} \cap \dots \cap U_{i_p}$ for $p \in \mathbb{N}$ and $i_0, \dots, i_p \in I$ and define $C^p(\mathcal{M}, \mathcal{F})$ as the set consisting of all tuples $\varphi = (\varphi_{i_0, \dots, i_p})_{i_0, \dots, i_p} \in \prod_{(i_0, \dots, i_p) \in I^{p+1}} \Gamma(U_{i_0, \dots, i_p}, \mathcal{F})$ which are antisymmetric with respect to the indices i_0, \dots, i_p . For $\varphi \in C^p(\mathcal{M}, \mathcal{F})$ we define $\delta_p \varphi = ((\delta_p \varphi)_{i_0, \dots, i_{p+1}})_{i_0, \dots, i_{p+1}} \in C^{p+1}(\mathcal{M}, \mathcal{F})$ as

$$(\delta_p \varphi)_{i_0, \dots, i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \varphi_{i_0, \dots, \widehat{i_j}, \dots, i_{p+1}}|_{U_{i_0, \dots, i_{p+1}}},$$

where the hat mark on \hat{i}_j means that the index i_j is omitted. Since $\delta_{p+1} \circ \delta_p = 0$, we have the ensuing complex

$$0 \longrightarrow C^0(\mathcal{M}, \mathcal{F}) \xrightarrow{\delta_0} C^1(\mathcal{M}, \mathcal{F}) \xrightarrow{\delta_1} C^2(\mathcal{M}, \mathcal{F}) \xrightarrow{\delta_2} \dots$$

The p -th cohomology group of the above complex is defined as

$$H^p(\mathcal{M}, \mathcal{F}) = Z^p(\mathcal{M}, \mathcal{F}) / B^p(\mathcal{M}, \mathcal{F}),$$

where $Z^p(\mathcal{M}, \mathcal{F}) = \ker \delta_p$ and $B^p(\mathcal{M}, \mathcal{F}) = \text{Im } \delta_{p-1}$ ($B^0(\mathcal{M}, \mathcal{F}) = \{0\}$).

Let $I' \subseteq I$ and set $\mathcal{M}' = \{U_i | i \in I'\}$. By restricting the indices of an element of $C^p(\mathcal{M}, \mathcal{F})$ to I' , we can define a restriction mapping $C^p(\mathcal{M}, \mathcal{F}) \rightarrow C^p(\mathcal{M}', \mathcal{F})$. We write $C^p(\mathcal{M}, \mathcal{M}', \mathcal{F})$ for the kernel of this mapping and define $H^p(\mathcal{M}, \mathcal{M}', \mathcal{F})$ to be the p -th cohomology group of the complex

$$0 \longrightarrow C^0(\mathcal{M}, \mathcal{M}', \mathcal{F}) \xrightarrow{\delta_0} C^1(\mathcal{M}, \mathcal{M}', \mathcal{F}) \xrightarrow{\delta_1} \dots$$

We have the following complex of short exact sequences

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C^0(\mathcal{M}, \mathcal{M}', \mathcal{F}) & \longrightarrow & C^0(\mathcal{M}, \mathcal{F}) & \longrightarrow & C^0(\mathcal{M}', \mathcal{F}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C^1(\mathcal{M}, \mathcal{M}', \mathcal{F}) & \longrightarrow & C^1(\mathcal{M}, \mathcal{F}) & \longrightarrow & C^1(\mathcal{M}', \mathcal{F}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

which yields the long exact sequence of cohomology groups [117, Thm. B.2.1]

$$\begin{aligned}
0 &\longrightarrow H^0(\mathcal{M}, \mathcal{M}', \mathcal{F}) \longrightarrow H^0(\mathcal{M}, \mathcal{F}) \longrightarrow H^0(\mathcal{M}', \mathcal{F}) \\
&\longrightarrow H^1(\mathcal{M}, \mathcal{M}', \mathcal{F}) \longrightarrow H^1(\mathcal{M}, \mathcal{F}) \longrightarrow H^1(\mathcal{M}', \mathcal{F}) \\
&\longrightarrow H^2(\mathcal{M}, \mathcal{M}', \mathcal{F}) \longrightarrow \dots
\end{aligned}
\tag{5.4.1}$$

5.4.2 The scalar-valued case

Theorem 5.4.1. *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $\mathcal{M} = \{\Omega_i \mid i \in I\}$ be an open covering of Ω . Then, $H^1(\mathcal{M}, \mathcal{E}^*) = 0$. Explicitly, this means that the sequence*

$$0 \longrightarrow \mathcal{E}^*(\Omega) \longrightarrow \prod \mathcal{E}^*(\Omega_i) \xrightarrow{\delta} Z^1(\mathcal{M}, \mathcal{E}^*) \longrightarrow 0
\tag{5.4.2}$$

is exact, where $Z^1(\mathcal{M}, \mathcal{E}^)$ consists of all tuples $(\varphi_{i,j})_{i,j} \in \prod \mathcal{E}^*(\Omega_{i,j})$ satisfying the cocycle condition*

$$\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} = 0 \quad \text{on } \Omega_{i,j,k}$$

for all $i, j, k \in I$ and

$$\delta = \delta_0 : \prod \mathcal{E}^*(\Omega_i) \rightarrow Z^1(\mathcal{M}, \mathcal{E}^*) : (\varphi_i)_i \rightarrow ((\varphi_j - \varphi_i)|_{\Omega_{i,j}})_{i,j}.$$

We shall prove this theorem in several steps.

Lemma 5.4.2. *Let $h_j \in \mathfrak{R}$ and let M_p be a weight sequence satisfying (M.1), (M.2)', and (NA). Then, there is $h'_j \in \mathfrak{R}$ with $h'_j \preceq h_j$ such that the weight sequence $M_p / \prod_{j=0}^p h'_j$ also satisfies (M.1), (M.2)', and (NA).*

Proof. By Lemma 2.2.2(ii) we first find $k_j \in \mathfrak{R}$ such that $p! \subset M_p / \prod_{j=0}^p k_j$. The sequence $h'_j \in \mathfrak{R}$ with $h'_0 = \min\{h_0, \sqrt{k_0}\}$, $h'_1 = \min\{h_1, \sqrt{k_1}\}$, and

$$h'_j = \min \left\{ h_j, \frac{m_j}{m_{j-1}} h'_{j-1}, \sqrt{k_j} \right\}, \quad j \geq 2,$$

satisfies all requirements. \square

Proposition 5.4.3. *Let $K_1, K_2 \subseteq \mathbb{R}^d$. Then, the sequence*

$$\begin{aligned} 0 &\longrightarrow \mathcal{E}^*[K_1 \cup K_2] \longrightarrow \mathcal{E}^*[K_1] \times \mathcal{E}^*[K_2] \\ &\longrightarrow \mathcal{E}^*[K_1 \cap K_2] \longrightarrow 0 \end{aligned}$$

is exact.

Proof. Roumieu case. This follows by dualizing the exact sequence from Proposition 2.2.14.

Beurling case. We only need to show the exactness at $\mathcal{E}^{(M_p)}[K_1 \cap K_2]$, the rest is obvious. Let $\varphi \in \mathcal{E}^{(M_p)}[K_1 \cap K_2]$. Lemma 2.2.2(ii) implies that there is $h_j \in \mathfrak{R}$ such that $\varphi \in \mathcal{E}^{\{N_p\}}[K_1 \cap K_2]$, where $N_p = M_p / \prod_{j=0}^p h_j$. By Lemma 5.4.2 we may assume that N_p satisfies (M.1), (M.2)', and (NE). The result now follows from the Roumieu case. \square

Proof of Theorem 5.4.1. STEP I: $I = \{1, 2\}$. Let $(K_{i,N})_N$ be an exhaustion by compact sets of Ω_i , $i = 1, 2$. We define the projective spectra $\mathcal{X} = (\mathcal{E}^*[K_{1,N} \cup K_{2,N}])_N$, $\mathcal{Y} = (\mathcal{E}^*[K_{1,N}] \times \mathcal{E}^*[K_{2,N}])_N$, and $\mathcal{Z} = (\mathcal{E}^*[K_{1,N} \cap K_{2,N}])_N$. Proposition 5.4.3 yields that the sequence of projective spectra

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

is exact. Since $\text{Proj}^0 \mathcal{X} \cong \mathcal{E}^*(\Omega_1 \cup \Omega_2)$, $\text{Proj}^0 \mathcal{Y} \cong \mathcal{E}^*(\Omega_1) \times \mathcal{E}^*(\Omega_2)$, and $\text{Proj}^0 \mathcal{Z} \cong \mathcal{E}^*(\Omega_1 \cap \Omega_2)$, it suffices to show that $\text{Proj}^1 \mathcal{X} = 0$.

Roumieu case: Follows from Theorem 2.3.7 and Proposition 5.2.1.

Beurling case: Let $(\Omega_{i,N})_N$ be an exhaustion by relatively compact open sets of Ω_i , $i = 1, 2$. The spectrum \mathcal{X} is equivalent to $\mathcal{X}_0 = (\mathcal{E}^{(M_p)}(\Omega_{1,N} \cup \Omega_{2,N}))_N$ and, thus, $\text{Proj}^1 \mathcal{X} \cong \text{Proj}^1 \mathcal{X}_0$ by Lemma 2.3.5. Since the spectrum \mathcal{X}_0 consists of Fréchet spaces and is reduced, the result follows from Lemma 2.3.6.

STEP II: *I is finite.* Follows from STEP I by using an induction argument as in the proof of the classical Mittag-Leffler lemma (see e.g. [117, Thm. 2.3.1]).

STEP III: *I is arbitrary.* Since \mathbb{R}^d is second countable, we may assume without loss of generality that I is countable (set $I = \mathbb{N}$) and that $\Omega_i \Subset \Omega$ for all $i \in \mathbb{N}$. We define the projective spectra $\mathcal{X} = \left(\mathcal{E}^* \left(\bigcup_{i=0}^N \Omega_i \right) \right)_N$, $\mathcal{Y} = \left(\prod_{i=0}^N \mathcal{E}^*(\Omega_i) \right)_N$, and $\mathcal{Z} = (Z^1(\mathcal{M}_N, \mathcal{E}^*))_N$, where $\mathcal{M}_N = \{\Omega_i \mid i = 0, \dots, N\}$. By STEP II, the sequence of projective spectra

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

is exact. Notice that $\text{Proj}^0 \mathcal{X} \cong \mathcal{E}^*(\Omega)$, $\text{Proj}^0 \mathcal{Y} \cong \prod \mathcal{E}^*(\Omega_i)$, and $\text{Proj}^0 \mathcal{Z} \cong Z^1(\mathcal{M}, \mathcal{E}^*)$. Hence, it suffices to show that $\text{Proj}^1 \mathcal{X} = 0$.

Roumieu case: Let $(K_N)_N$ be an exhaustion by compact sets of Ω . The spectrum \mathcal{X} is equivalent to $\mathcal{X}_0 = (\mathcal{E}^{\{M_p\}}[K_N])_N$ and, thus, $\text{Proj}^1 \mathcal{X} \cong \text{Proj}^1 \mathcal{X}_0$ by Lemma 2.3.5. The result now follows from Theorem 2.3.7 and Proposition 5.2.1.

Beurling case: Follows directly from Lemma 2.3.6. \square

We end this section by discussing the topological exactness of the sequence (5.4.2). We endow $\prod \mathcal{E}^*(\Omega_i)$ with the product topology and $Z^1(\mathcal{M}, \mathcal{E}^*)$ with the relative topology induced by $\prod \mathcal{E}^*(\Omega_{i,j})$ (endowed with the product topology). Observe that $Z^1(\mathcal{M}, \mathcal{E}^*)$ is a closed subspace of $\prod \mathcal{E}^*(\Omega_{i,j})$ and that the mapping δ is continuous.

Proposition 5.4.4. *The sequence (5.4.2) is topologically exact if I is countable.*

In the Beurling case, this is a consequence of the open mapping theorem. We need two preparatory lemmas for the Roumieu case.

Lemma 5.4.5. *Let X be a topological space and let \mathcal{F} be a sheaf on X . Suppose that $H^1(\mathcal{M}, \mathcal{F}) = 0$ for all finite open coverings \mathcal{M} . Then, $H^p(\mathcal{M}, \mathcal{F}) = 0$ for all $p \geq 1$ and all finite open coverings \mathcal{M} .*

Proof. We use induction on $N = |\mathcal{M}|$. The case $N = 1$ is clear. Suppose that the result holds for N and let us prove it for $N+1$. Let $\mathcal{M} = \{U_i \mid i = 0, \dots, N\}$ be an arbitrary open covering and define $\mathcal{M}' = \{U_i \mid i = 0, \dots, N-1\}$. The induction hypothesis implies that $H^p(\mathcal{M}', \mathcal{F}) = 0$ for all $p \geq 1$. Therefore, the long exact sequence of cohomology groups (5.4.1) yields that $H^p(\mathcal{M}, \mathcal{F}) \cong H^p(\mathcal{M}, \mathcal{M}', \mathcal{F})$ for all $p \geq 2$. Observe that $C^p(\mathcal{M}, \mathcal{M}', \mathcal{F}) \cong C^{p-1}(\widetilde{\mathcal{M}}, \mathcal{F})$ for all $p \geq 1$, where $\widetilde{\mathcal{M}} = \{U_i \cap U_n \mid i = 0, \dots, N-1\}$. Hence, $H^p(\mathcal{M}, \mathcal{F}) \cong H^p(\mathcal{M}, \mathcal{M}', \mathcal{F}) \cong H^{p-1}(\widetilde{\mathcal{M}}, \mathcal{F}) = 0$ for all $p \geq 2$, where we have used the induction hypothesis once again in the last equality. \square

Lemma 5.4.6. *Let*

$$0 \longrightarrow X_0 \xrightarrow{\delta_0} X_1 \xrightarrow{\delta_1} \dots \xrightarrow{\delta_{N-1}} X_N \longrightarrow 0$$

be an exact sequence of ultrabornological (PLS)-spaces. Then, the sequence is automatically topologically exact and $\ker \delta_j$ is an ultrabornological (PLS)-space for each $j = 0, \dots, N-1$.

Proof. This is a consequence of De Wilde's open mapping theorem and the fact that a closed subspace A of an ultrabornological (PLS)-space X is ultrabornological if and only if X/A is complete [57, Cor. 1.4]. \square

Proof of Proposition 5.4.4 (Roumieu case). Throughout this proof we shall repeatedly use that the class of (PLS)-spaces is closed under taking countable products and closed subspaces [57, Prop.

1.2] and that the class of ultrabornological spaces is closed under taking countable products and quotients.

STEP I: *I is finite.* Suppose that $I = \{0, \dots, N\}$ for some $N \in \mathbb{N}$. Theorem 5.4.1 and Lemma 5.4.5 imply that the sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{E}^{\{M_p\}}(\Omega) &\longrightarrow \prod \mathcal{E}^{\{M_p\}}(\Omega_i) \longrightarrow C^1(\mathcal{M}, \mathcal{E}^{\{M_p\}}) \\ &\longrightarrow \dots \longrightarrow C^N(\mathcal{M}, \mathcal{E}^{\{M_p\}}) \longrightarrow 0 \end{aligned}$$

is exact. Notice that $C^p(\mathcal{M}, \mathcal{E}^{\{M_p\}})$ is isomorphic to a finite product of spaces of the form $\mathcal{E}^{\{M_p\}}(\Omega_{i_0, \dots, i_p})$, $0 \leq i_0 < i_1 < \dots < i_p \leq N$. We endow these spaces with the product topology. The result now follows from Proposition 5.2.1 and Lemma 5.4.6.

STEP II: *I is countably infinite.* Set $I = \mathbb{N}$. Consider the projective spectra defined in STEP III of the proof of Theorem 5.4.1. By STEP I the complex

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

consists of short topologically exact sequences. Since every (PLS) -space E has a strict ordered web and $\text{Proj}^1 \mathcal{X} = 0$ (see STEP III of the proof of Theorem 5.4.1), [164, Thm. 3.3] implies that the mapping δ appearing in (5.4.2) is a topological homomorphism. Hence, we obtain that $Z^1(\mathcal{M}, \mathcal{E}^{\{M_p\}})$ is an ultrabornological (PLS) -space from Proposition 5.2.1. The result now follows from Lemma 5.4.6. \square

5.4.3 The vector-valued case

Theorem 5.4.7. *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $\mathcal{M} = \{\Omega_i \mid i \in I\}$ be an open covering of Ω . Then, $H^1(\mathcal{M}, \mathcal{E}^*(\cdot; F)) = 0$ in the following cases:*

(i) $*$ = (M_p) and F a Fréchet space.

(ii) $*$ = $\{M_p\}$ and F a (DFS)-space such that F' satisfies (DN).

(iii) $*$ = $\{p!\}$ and F a (PLS)-space satisfying the dual interpolation estimate for small theta.

Proof. As in the scalar-valued case, $H^1(\mathcal{M}, \mathcal{E}^*(\cdot; F)) = 0$ means that

$$0 \longrightarrow \mathcal{E}^*(\Omega; F) \longrightarrow \prod \mathcal{E}^*(\Omega_i; F) \xrightarrow{\delta_F} Z^1(\mathcal{M}, \mathcal{E}^*(\cdot, F)) \longrightarrow 0$$

is exact, where $Z^1(\mathcal{M}, \mathcal{E}^*(\cdot, F))$ consists of all tuples $(\varphi_{i,j})_{i,j} \in \prod \mathcal{E}^*(\Omega_{i,j}; F)$ satisfying the cocycle condition

$$\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} = 0 \quad \text{on } \Omega_{i,j,k}$$

for all $i, j, k \in I$ and

$$\delta_F = \prod \mathcal{E}^*(\Omega_i; F) \rightarrow Z^1(\mathcal{M}, \mathcal{E}^*(\cdot, F)) : (\varphi_i)_i \rightarrow ((\varphi_j - \varphi_i)|_{\Omega_{i,j}})_{i,j}.$$

We only need to show that δ_F is surjective, the rest is clear. Since \mathbb{R}^d is second countable, we may assume that I is countable. Proposition 5.3.4 and [92, Prop. 1.5] yield the following isomorphisms of locally convex spaces: $\mathcal{E}^*(\Omega; F) \cong \mathcal{E}^*(\Omega)_{\varepsilon F}$, $\prod_{i \in I} \mathcal{E}^*(\Omega_i; F) \cong (\prod_{i \in I} \mathcal{E}^*(\Omega_i))_{\varepsilon F}$, and $Z^1(\mathcal{M}, \mathcal{E}^*(\cdot, F)) \cong Z^1(\mathcal{M}, \mathcal{E}^*)_{\varepsilon F}$. Furthermore, notice that $\delta_F = \delta \varepsilon \text{id}_F$.

(i) The spaces $\prod_{i \in I} \mathcal{E}^{(M_p)}(\Omega_i)$ and $Z^1(\mathcal{M}, \mathcal{E}^{(M_p)})$ are nuclear Fréchet spaces. By the above remarks the result therefore follows from Theorem 5.4.1 and the ensuing general result [156, Exercise 45.3]: Let $T_1 : X_1 \rightarrow Y_1$ and $T_2 : X_2 \rightarrow Y_2$ be two surjective continuous linear mappings between Fréchet spaces. Then, the mapping $T_1 \widehat{\otimes}_{\pi} T_2 : X_1 \widehat{\otimes}_{\pi} X_2 \rightarrow Y_1 \widehat{\otimes}_{\pi} Y_2$ is also surjective.

(ii) and (iii) This follows from Theorem 5.4.1, Proposition 5.3.4, and Lemma 5.3.1; the assumptions of Lemma 5.3.1 are satisfied by Proposition 5.4.4 and Proposition 5.3.8 (Corollary 5.3.10 in the real analytic case). \square

Chapter 6

Optimal embeddings of spaces of infrahyperfunctions into differential algebras

6.1 Introduction

Let M_p be a weight sequence satisfying $(M.1)$, $(M.2)$, $(M.2)^*$, (QA) , and (NE) . The aim of this section is to show that it is possible to embed the space of infrahyperfunctions of class $\{M_p\}$ into the algebra of generalized functions of class $\{M_p\}$ (introduced in Section 3.4) in such a way that the multiplication of ultradifferentiable functions of class $\{M_p\}$ is preserved. In view of the impossibility result presented in Section 3.5, such an embedding is optimal. Colombeau algebras containing the space of hyperfunctions on the unit circle were considered in [157, 46, 47] but the preservation of multiplication of all real analytic functions was not achieved there; we resolved this issue in [36]. To the best of our knowledge, no prior work has been done in the general local case.

Let us briefly comment on the methods to be employed in this chapter. Firstly, as pointed out in Section 3.4, it is from the outset not clear whether $\mathcal{G}^{\{M_p\}}$ is a sheaf on \mathbb{R}^d . To show that this is indeed the case, we shall use the solution to the first Cousin problem for vector-valued quasianalytic functions obtained in Chapter 5. We are inspired by the theory of hyperfunctions in one dimension: The fact that the hyperfunctions form a sheaf on \mathbb{R} is a direct consequence of the classical Mittag-Leffler lemma (= solution to the first Cousin problem for the sheaf of holomorphic functions of one complex variable) [117, Thm. 3.2.3]. Similarly, one can verify that the solvability of the first Cousin problem for the sheaf $\mathcal{E}_{\mathcal{N}}^{\{M_p\}}$ consisting of the spaces of negligible sequences of ultradifferentiable functions of class $\{M_p\}$ would imply that $\mathcal{G}^{\{M_p\}}$ is a sheaf on \mathbb{R}^d . The fundamental observation is then that $\mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega) = \mathcal{E}^{\{M_p\}}(\Omega; s^{\{M_p\}})$ for each open subset Ω of \mathbb{R}^d , which enables us to use Theorem 5.4.7 to solve the Cousin problem for the sheaf $\mathcal{E}_{\mathcal{N}}^{\{M_p\}}$; we refer to Section 6.3 for more details. Secondly, we discuss the mollifier sequences to be employed in our embedding of $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$ into $\Gamma_c(\mathbb{R}^d, \mathcal{G}^{\{M_p\}})$. Recall that, in the non-quasianalytic case (Chapter 4), we used mollifier sequences of the form $(n^d \rho(n \cdot))_n$, where ρ is the inverse Fourier transform of a sufficiently regular smooth compactly supported function χ that is equal to 1 in a neighbourhood of the origin. Due to the absence of non-trivial compactly supported quasianalytic functions, one can no longer expect that such mollifier sequences will do the job in the present setting. Our idea is to replace the single cut-off function χ by an *analytic cut-off sequence*, i.e. a sequence of compactly supported smooth functions satisfying adequate bounds for derivatives only up to a certain order; see Section 6.2 for the precise definition. Such sequences were introduced by Hörmander in [79] to give a description of the analytic wave front set of a distribution that resembles the definition of the classical smooth wave front set [79, Sect. 8.4.2]. Later on, analytic cut-off sequences also played an important role in his paper on infrahyperfunctions [78].

In our opinion, the idea to overcome the problem that there are no non-zero compactly supported quasianalytic functions by replacing a single cut-off function by a sequence of suitable cut-off functions is truly ingenious; we are very much indebted to Hörmander, as a lot of the techniques used in this chapter are modifications of his ideas from [78].

This chapter is organized as follows. In the preliminary Section 6.2, we collect some basic facts concerning analytic cut-off sequences. The fact that $\mathcal{G}^{\{M_p\}}$ is a soft sheaf on \mathbb{R}^d is shown in Section 6.3. Finally, in Section 6.4, we follow the method presented in Section 3.3 to embed the sheaf of infrahyperfuctions of class $\{M_p\}$ into the sheaf of algebras of generalized functions of class $\{M_p\}$ and to show that the multiplication of ultradifferentiable functions of class $\{M_p\}$ is preserved under this embedding.

6.2 Analytic cut-off sequences

In this section, we define analytic cut-off sequences and give some of their basic properties.

Let $\Omega \subseteq \mathbb{R}^d$ be open. A sequence $(\kappa_n)_n$ in $\mathcal{D}(\Omega)$ is said to be an analytic cut-off sequence (supported in Ω) [79, 78] if

- $0 \leq \kappa_n \leq 1$ for all $n \in \mathbb{N}$.
- $(\kappa_n)_n$ is a bounded sequence in $\mathcal{D}(\Omega)$.
- There is $L \geq 1$ such that

$$\|\partial^\alpha \kappa_n\|_{L^\infty} \leq L(Ln)^{|\alpha|}, \quad n \in \mathbb{N}, \quad |\alpha| \leq n.$$

Let $K \Subset \Omega$. We call $(\kappa_n)_n$ an analytic cut-off sequence for K if there is an open neighbourhood Ω' of K such that $\kappa_n \equiv 1$ on Ω' for all $n \in \mathbb{N}$. In [79, Thm. 1.4.2], it is shown that for every $K \Subset \mathbb{R}^d$ and every open neighbourhood Ω of K there is an analytic cut-off

sequence for K supported in Ω . The following analogue of Lemma 2.2.9 holds for analytic cut-off sequences.

Lemma 6.2.1. (Proof of [78, Thm. 3.4]) *Let M_p be a weight sequence satisfying (M.1) and (NE) and let $h > 0$. Let Ω be a relatively compact open subset of \mathbb{R}^d and let $(\kappa_n)_n$ be an analytic cut-off sequence supported in Ω . Then,*

$$|\xi|^n |\widehat{\kappa_n \varphi}(\xi)| \leq CL|\Omega| \|\varphi\|_{\mathcal{E}^{M_p, h}(\overline{\Omega})} (\sqrt{d}(h + Lk))^n M_n, \quad \xi \in \mathbb{R}^d,$$

for all $n \in \mathbb{N}$ and $\varphi \in \mathcal{E}^{M_p, h}(\overline{\Omega})$, where $C, k > 0$ are chosen in such a way that $p^p \leq Ck^p M_p$ for all $p \in \mathbb{N}$.

We now state an extension procedure that will be of crucial importance in this chapter. We need some preparation. Let M_p be a weight sequence. For $h > 0$ we write $\mathcal{B}^{M_p, h}(\mathbb{R}^d)$ for the Banach space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|} M_\alpha} < \infty.$$

As customary, we denote by m the counting function of the sequence $(m_p)_{p \geq 1}$, that is,

$$m(t) = \sum_{m_p \leq t} 1, \quad t \geq 0.$$

If M_p satisfies (M.1), then the following equality holds

$$\frac{t^{m(t)}}{M_{m(t)}} = \sup_{p \in \mathbb{N}} \frac{t^p M_0}{M_p} = e^{M(t)}, \quad t \geq 0.$$

Lemma 6.2.2. (cf. first part of the proof of [78, Thm. 3.4]) *Let M_p be a weight sequence satisfying (M.1), (M.2), and (NE) and let $h > 0$. Let Ω be a relatively compact open subset of \mathbb{R}^d and let $(\kappa_p)_p$ be an analytic cut-off sequence supported in Ω . Let $r \geq 1$ and let $(\psi_n)_n$ be a uniformly bounded sequence of continuous functions on \mathbb{R}^d such that $\text{supp } \psi_0 \subseteq \overline{B}(0, r)$ and*

$$\text{supp } \psi_n \subseteq \overline{B}(0, rn) \setminus B(0, n-1), \quad n \geq 1.$$

Then, there are a sequence $(p_n)_n$ of natural numbers and $k > 0$ such that

$$R(\varphi) := \sum_{n=0}^{\infty} (\kappa_{p_n} \varphi) * \mathcal{F}^{-1}(\psi_n) \in \mathcal{B}^{M_p, k}(\mathbb{R}^d),$$

for all $\varphi \in \mathcal{E}^{M_p, h}(\overline{\Omega})$. Moreover, the convergence of the series $R(\varphi)$ holds in the topology of $\mathcal{B}^{M_p, k}(\mathbb{R}^d)$ and the mapping $R : \mathcal{E}^{M_p, h}(\overline{\Omega}) \rightarrow \mathcal{B}^{M_p, k}(\mathbb{R}^d)$ is continuous.

Proof. By Lemma 6.2.1 there are $C, k_0 > 0$ such that

$$|\xi|^p |\widehat{\kappa_p \varphi}(\xi)| \leq C \|\varphi\|_{\mathcal{E}^{M_p, h}(\overline{\Omega})} k_0^p M_p, \quad \xi \in \mathbb{R}^d, p \in \mathbb{N},$$

for all $\varphi \in \mathcal{E}^{M_p, h}(\overline{\Omega})$. For $p = m(t)$, $t \geq 0$, and $k_0 t \leq |\xi| \leq 2rk_0 t$ we obtain that

$$|\widehat{\kappa_{m(t)} \varphi}(\xi)| \leq C \|\varphi\|_{\mathcal{E}^{M_p, h}(\overline{\Omega})} e^{-M(\xi/(2k_0 r))}.$$

Set $p_0 = 0$ and $p_n = m((n-1)/k_0)$ for $n \geq 1$. Hence, for $n \geq k_0 + 1$ and $(n-1) \leq |\xi| \leq rn$, it holds that

$$|\widehat{\kappa_{p_n} \varphi}(\xi)| \leq C \|\varphi\|_{\mathcal{E}^{M_p, h}(\overline{\Omega})} e^{-M(\xi/(2k_0 r))}.$$

Choose $C' > 0$ such that $\|\psi_n\|_{L^\infty} \leq C'$ for all $n \in \mathbb{N}$. Set $k = 2H^2 k_0 r$. We have that

$$\begin{aligned} & \int_{\mathbb{R}^d} \left| \mathcal{F} \left(\sum_{n \geq k_0+1}^{\infty} (\kappa_{p_n} \varphi) * \mathcal{F}^{-1}(\psi_n) \right) (\xi) \right| e^{M(\xi/k)} d\xi \\ & \leq \sum_{n \geq k_0+1}^{\infty} \int_{\mathbb{R}^d} |\widehat{\kappa_{p_n} \varphi}(\xi)| |\psi_n(\xi)| e^{M(\xi/k)} d\xi \\ & \leq C C' \|\varphi\|_{\mathcal{E}^{M_p, h}(\overline{\Omega})} \sum_{n \geq k_0+1}^{\infty} \int_{(n-1) \leq |\xi| \leq rn} e^{M(\xi/k) - M(\xi/(2k_0 r))} d\xi \\ & \leq C'' \|\varphi\|_{\mathcal{E}^{M_p, h}(\overline{\Omega})}, \end{aligned}$$

where

$$C'' = C_0^2 C C' \sum_{n=k_0}^{\infty} e^{-M(n/k)} \int_{\mathbb{R}^d} e^{-M(\xi/(2Hk_0 r))} d\xi < \infty.$$

On the other hand, we have that

$$\begin{aligned} & \int_{\mathbb{R}^d} |\mathcal{F}((\kappa_{p_n} \varphi) * \mathcal{F}^{-1}(\psi_n))(\xi)| e^{M(\xi/k)} d\xi \\ & \leq C' e^{M(rk_0/k)} |B(0, rk_0)| \|\Omega\| \|\varphi\|_{C(\overline{\Omega})}, \end{aligned}$$

for all $n \leq k_0$. The result now follows from Lemma 2.2.7. \square

6.3 Sheaf properties

Let M_p be a weight sequence satisfying (M.1), (M.2), (M.2)*, (QA), and (NE). We now prove that $\mathcal{G}^{\{M_p\}}$ is a soft sheaf. As pointed out in Section 3.4, it is clear that the presheaf $\mathcal{G}^{\{M_p\}}$ satisfies (S1). To show that it also satisfies (S2), we follow the method described in the introduction of this chapter (Section 6.1).

We start by observing that the projective descriptions of the spaces $\mathcal{E}^{\{M_p\}}(\Omega)$ and $s^{\{M_p\}}$ (Theorem 5.2.3 and Example 3.2.2) imply that

$$\mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega) =: s^{\{M_p\}}(\mathcal{E}^{\{M_p\}}(\Omega)) = \mathcal{E}^{\{M_p\}}(\Omega; s^{\{M_p\}}).$$

Furthermore, $s^{\{M_p\}}$ is a (DFS)-space and its strong dual $s'^{\{M_p\}}$ satisfies (DN) if and only if M_p satisfies (M.2)* [103, Prop. 4.1]. Hence, Theorem 5.4.7 yields that:

Corollary 6.3.1. *Let M_p be a weight sequence satisfying (M.1), (M.2), (M.2)*, (QA), and (NE). Let $\Omega \subseteq \mathbb{R}^d$ be open and let $\mathcal{M} = \{\Omega_i \mid i \in I\}$ be an open covering of Ω . Suppose that $(\varphi_{i,j,n})_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega_i \cap \Omega_j)$, $i, j \in I$, satisfy*

$$\varphi_{i,j,n} + \varphi_{j,k,n} + \varphi_{k,i,n} = 0 \quad \text{on } \Omega_i \cap \Omega_j \cap \Omega_k$$

for all $i, j, k \in I$ and $n \in \mathbb{N}$. Then, there are $(\varphi_{i,n})_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega_i)$, $i \in I$, such that

$$\varphi_{i,j,n} = \varphi_{j,n} - \varphi_{i,n} \quad \text{on } \Omega_i \cap \Omega_j$$

for all $i, j \in I$ and $n \in \mathbb{N}$.

Proposition 6.3.2. *Let M_p be a weight sequence satisfying (M.1), (M.2), $(M.2)^*$, (QA), and (NE). Then, $\mathcal{G}^{\{M_p\}}$ is a soft sheaf.*

Proof. Firstly, we show that $\mathcal{G}^{\{M_p\}}$ satisfies (S2). Let $\Omega \subseteq \mathbb{R}^d$ be open and let $(\Omega_i)_i$ be an open covering of Ω . Suppose that $f_i = [(\varphi_{i,n})_n] \in \mathcal{G}^{\{M_p\}}(\Omega_i)$ are given such that $f_i = f_j$ on $\Omega_i \cap \Omega_j$, for all i, j . This means that there are $(\psi_{i,j,n})_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega_i \cap \Omega_j)$ such that $\psi_{i,j,n} = \varphi_{j,n} - \varphi_{i,n}$ on $\Omega_i \cap \Omega_j$ for all i, j and $n \in \mathbb{N}$. Corollary 6.3.1 implies that there are $(\psi_{i,n})_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega_i)$ such that $\psi_{i,j,n} = \psi_{j,n} - \psi_{i,n}$ on $\Omega_i \cap \Omega_j$ for all i, j and $n \in \mathbb{N}$. Therefore, the function φ_n given by $\varphi_n(x) = \varphi_{i,n}(x) - \psi_{i,n}(x)$ for $x \in \Omega_i$ is a well-defined element of $\mathcal{E}^{\{M_p\}}(\Omega)$. Furthermore, we have that $(\varphi_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega)$ and that $f = [(\varphi_n)_n] \in \mathcal{G}^{\{M_p\}}(\Omega)$ satisfies $f|_{\Omega_i} = f_i$ for all i . Next, we prove that $\mathcal{G}^{\{M_p\}}$ is soft. Fix $K \Subset \mathbb{R}^d$. Let Ω be an arbitrary open neighbourhood of K and choose a relatively compact open set Ω' in Ω such that $K \Subset \Omega'$. It suffices to show that for every $f = [(\varphi_n)_n] \in \mathcal{G}^{\{M_p\}}(\Omega)$ there is $g = [(\psi_n)_n] \in \mathcal{G}^{\{M_p\}}(\mathbb{R}^d)$ such that $g = f$ on Ω' . Let $(\kappa_p)_p$ be an analytic cut-off sequence for $\overline{\Omega'}$ supported in Ω . Lemma 6.2.1 implies that there are $C, h > 0$ such that

$$|\xi|^p |\widehat{\kappa_p \varphi_n}(\xi)| \leq C e^{M(n)} h^p M_p, \quad \xi \in \mathbb{R}^d,$$

for all $p, n \in \mathbb{N}$. Let $\chi \in \mathcal{D}(\mathbb{R}^d)$ with $\text{supp } \chi \subseteq \overline{B}(0, 2)$ be such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $\overline{B}(0, 1)$. We define $\chi_0 = \chi$ and

$$\chi_n = \chi \left(\frac{\cdot}{H^2 h n} \right), \quad n \geq 1.$$

Set

$$\psi_n = (\kappa_{p_n} \varphi_n) * \mathcal{F}^{-1}(\chi_n), \quad n \in \mathbb{N},$$

where $p_n = m(Hn) + d + 1$. We first show that $(\varphi_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\mathbb{R}^d)$. Let $\lambda > 0$ be arbitrary. Since the sequence $(\kappa_p)_p$ is bounded in $\mathcal{D}(\Omega)$, it holds that

$$|\widehat{\kappa_p \varphi_n}(\xi)| \leq \frac{C' e^{M(\lambda n/H)}}{(1 + |\xi|)^{d+1}}, \quad \xi \in \mathbb{R}^d,$$

for all $p, n \in \mathbb{N}$ and some $C' > 0$. Set $k = (2H^3h)/\lambda$. Then,

$$\int_{\mathbb{R}^d} |\widehat{\psi}_n(\xi)| e^{M(\xi/k)} d\xi = \int_{\mathbb{R}^d} |\widehat{\kappa_{p_n} \varphi_n}(\xi)| \chi_n(\xi) e^{M(\xi/k)} d\xi \leq C'' e^{M(\lambda n)},$$

where

$$C'' = C_0 C' \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|)^{d+1}} d\xi < \infty.$$

The sequence $(\psi_n)_n$ is therefore moderate by Lemma 2.2.7. We still need to show that $(\psi_n - \varphi_n)_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega')$. To this end, we use Lemma 3.4.5. For $n \geq 1$, it holds that

$$\begin{aligned} & \sup_{x \in \Omega'} |\psi_n(x) - \varphi_n(x)| \\ & \leq \sup_{x \in \Omega'} |((\kappa_{p_n} \varphi_n) * \mathcal{F}^{-1}(\chi_n))(x) - (\kappa_{p_n} \varphi_n)(x)| \\ & \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{\kappa_{p_n} \varphi_n}(\xi)| (1 - \chi_n(\xi)) d\xi \\ & \leq \frac{C_0 C (Hh)^{d+1} M_{d+1} e^{M(n)}}{(2\pi)^d} \int_{|\xi| \geq H^2 h n} \frac{(Hh)^{m(Hn)} M_{m(Hn)}}{|\xi|^{m(Hn)+d+1}} d\xi \\ & \leq C' e^{-M(n)}, \end{aligned}$$

where

$$C' = \frac{C_0^2 C (Hh)^{d+1} M_{d+1}}{(2\pi)^d} \int_{|\xi| \geq H^2 h} \frac{1}{|\xi|^{d+1}} d\xi < \infty.$$

□

6.4 Construction of the embedding

We solve Problem 3.3.1 for the pair $(E, F) = (\mathfrak{B}^{\{M_p\}}, \mathcal{E}^{\{M_p\}})$ in this section. We start with a discussion about the mollifier sequences to be employed in our embedding of $\mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$ into $\Gamma_c(\mathbb{R}^d, \mathcal{G}^{\{M_p\}})$. We shall follow a similar approach as in Section 4.2 but use analytic cut-off sequences instead of a single cut-off sequence.

Let M_p be a weight sequence satisfying (NE) and let $(\chi_n)_n \subseteq \mathcal{D}(\mathbb{R}^d)$. The sequence $(\rho_n)_n$ with $\rho_n = n^d \mathcal{F}^{-1}(\chi_n)(n \cdot)$ is said to be an $\{M_p\}$ -mollifier sequence if

- χ_n is even for all $n \in \mathbb{N}$.
- $(\chi_n)_n$ is an analytic cut-off sequence for $\overline{B}(0, 1)$.
- for every $c > 0$ there are $C, h, \lambda > 0$ such that

$$\sup_{|x| \geq c} |\partial^\alpha \rho_n(x)| \leq C e^{-M(\lambda n)} h^{|\alpha|} M_\alpha, \quad \alpha \in \mathbb{N}^d, n \in \mathbb{N}. \quad (6.4.1)$$

The next lemma show the existence of such mollifier sequences; in fact, we provide an explicit construction of a $\{p!\}$ -mollifier sequence in its proof.

Lemma 6.4.1. *For every weight sequence M_p satisfying (NE) there exists an $\{M_p\}$ -mollifier sequence.*

Proof. It suffices to consider the case $M_p = p!$. Moreover, if $(\rho_n)_n$ is a one-dimensional $\{p!\}$ -mollifier sequence, then $(\boldsymbol{\rho}_n)_n$ with

$$\boldsymbol{\rho}_n = \underbrace{\rho_n \otimes \cdots \otimes \rho_n}_{d \text{ times}}$$

is a d -dimensional $\{p!\}$ -mollifier sequence. Hence, we may also assume that $d = 1$. We denote by H the characteristic function of $[-1, 1]$. Define

$$H_n = \left(\frac{n}{2}\right)^n \underbrace{H(n \cdot) * \cdots * H(n \cdot)}_{n \text{ times}}, \quad n \in \mathbb{N}.$$

Notice that $\text{supp } H_n \subseteq [-1, 1]$ and $\int_{\mathbb{R}} H_n(x) dx = 1$ for all $n \in \mathbb{N}$. Next, let $(\kappa_n)_n$ be an analytic cut-off sequence for $[-2, 2]$ consisting of even functions and set $\chi_n = \kappa_n * H_n$. Then, $(\chi_n)_n$ is an analytic cut-off sequence for $[-1, 1]$ consisting of even functions. We now show that additionally (6.4.1) is satisfied (for $M_p = p!$). Observe that

$$\rho_n(z) = n \mathcal{F}^{-1}(\chi_n)(nz) = n \mathcal{F}^{-1}(\kappa_n)(nz) (\text{sinc}(z))^n, \quad z \in \mathbb{C}, n \in \mathbb{N},$$

where, as customary, $\text{sinc}(z) = \sin z/z$. Let $0 < a < \pi$ be arbitrary. Then, $|\text{sinc}(x)| \leq \text{sinc}(a) =: b < 1$ for all $a \leq |x| \leq \pi$. Furthermore, $|\text{sinc}(x + iy)| \leq e^{|y|}/\pi$ for all $|x| \geq \pi$. Choose μ such that $\max\{b, 1/\pi\} < \mu < 1$. Then, there exists $r > 0$ such that $|\text{sinc}(x + iy)| \leq \mu$ for all $|x| \geq a$ and $|y| \leq r$. Since the sequence $(\kappa_n)_n$ is bounded in $\mathcal{D}(\mathbb{R})$, there are $C', \gamma > 0$ such that $|z| |\mathcal{F}^{-1}(\kappa_n)(z)| \leq C' e^{\gamma|y|}$ for all $z = x + iy \in \mathbb{C}$ and $n \in \mathbb{N}$. Choose $0 < r_0 < r$ so small that $e^{\gamma r_0} \mu < 1$. By combining the above two inequalities, we obtain that

$$|\rho_n(x + iy)| \leq \frac{C'(e^{\gamma r_0} \mu)^n}{a}$$

for all $|x| \geq a$, $|y| \leq r_0$, and $n \in \mathbb{N}$. Property (6.4.1) now follows from the Cauchy estimates. \square

Throughout the rest of this section we fix an $\{M_p\}$ -mollifier sequence $(\rho_n)_n$. Suppose that $\text{supp } \chi_n \subseteq \overline{B}(0, r)$ for all $n \in \mathbb{N}$ and some fixed $r > 0$. We are ready to embed $\mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$ into $\Gamma_c(\mathbb{R}^d, \mathcal{G}^{\{M_p\}})$. Notice that, since ρ_n is an entire function, the convolution $f * \rho_n = \langle f(x), \rho_n(\cdot - x) \rangle$ is well-defined for $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$.

Proposition 6.4.2. *Let M_p be a weight sequence satisfying (M.1), (M.2), and (NE). Then, the mapping*

$$\iota_c : \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d) \rightarrow \Gamma_c(\mathbb{R}^d, \mathcal{G}^{\{M_p\}}) : f \rightarrow [(f * \rho_n)_n]$$

is a linear embedding such that $\text{supp } \iota_c(f) = \text{supp } f$ for all $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$ and $\tilde{P}(D) \circ \iota_c = \iota_c \circ P(D)$ for all ultradifferential operators $P(D)$ of class $\{M_p\}$.

The proof of Proposition 6.4.2 is based on Lemma 6.2.2 and the ensuing proposition.

Proposition 6.4.3. *Let M_p be a weight sequence satisfying (M.1), (M.2), and (NE). Let $\Omega \subseteq \mathbb{R}^d$ be open and let Ω' be a relatively compact open subset of Ω . Let $\kappa \in \mathcal{D}(\Omega)$ be such that $0 \leq \kappa \leq 1$ and $\kappa \equiv 1$ in a neighbourhood of $\overline{\Omega'}$. Then, $((\kappa\varphi) * \rho_n - \varphi)_n \in \mathcal{E}_N^{\{M_p\}}(\Omega')$ for all $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega)$. In particular, $(\kappa\varphi) * \rho_n \rightarrow \varphi$ in $\mathcal{E}^{\{M_p\}}(\Omega')$.*

Proof. Let $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega)$ be arbitrary. We claim that

$$((\kappa\varphi) * \rho_n - (\kappa_n\varphi) * \rho_n)_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega')$$

for any bounded sequence $(\kappa_n)_n$ in $\mathcal{D}(\Omega)$ such that $0 \leq \kappa_n \leq 1$ and for which there is a neighbourhood Ω'' of $\overline{\Omega'}$ such that $\kappa_n \equiv 1$ in Ω'' for all $n \in \mathbb{N}$. We first prove that

$$((\kappa_n\varphi) * \rho_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\mathbb{R}^d). \quad (6.4.2)$$

Let $\lambda > 0$ be arbitrary. Since the sequence $(\kappa_n\varphi)_n$ is bounded in $\mathcal{D}(\mathbb{R}^d)$, there is $C > 0$ such that

$$|\widehat{\kappa_n\varphi}(\xi)| \leq \frac{C}{(1 + |\xi|)^{d+1}}, \quad \xi \in \mathbb{R}^d, n \in \mathbb{N}.$$

We have that

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{F}((\kappa_n\varphi) * \rho_n)(\xi)| e^{M(\lambda\xi/r)} d\xi &= \int_{\mathbb{R}^d} |\widehat{\kappa_n\varphi}(\xi)| \chi_n(\xi/n) e^{M(\lambda\xi/r)} d\xi \\ &\leq C e^{M(\lambda n)} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|)^{d+1}} d\xi \end{aligned}$$

and therefore the sequence is moderate by Lemma 2.2.7. Hence, by Lemma 3.4.5, it suffices to show that for every $K \Subset \Omega'$ there is $\lambda > 0$ such that

$$\sup_{n \in \mathbb{N}} \max_{x \in K} |((\kappa\varphi) * \rho_n)(x) - ((\kappa_n\varphi) * \rho_n)(x)| e^{M(\lambda n)} < \infty.$$

Choose $c > 0$ such that $\kappa - \kappa_n \equiv 0$ in $K + B(0, c)$ and $K' \Subset \Omega$ such that $\text{supp}(\kappa - \kappa_n) \subseteq K'$ for all $n \in \mathbb{N}$. Condition (6.4.1) implies that there are $C, \lambda > 0$

$$\begin{aligned} &\max_{x \in K} |((\kappa\varphi) * \rho_n)(x) - ((\kappa_n\varphi) * \rho_n)(x)| \\ &\leq \max_{x \in K} \left| \int_{\mathbb{R}^d} (\kappa(x-t) - \kappa_n(x-t)) \varphi(x-t) \rho_n(t) dt \right| \\ &\leq 2C \|\varphi\|_{C(K')} |K'| e^{-M(\lambda n)}. \end{aligned}$$

We now show the actual statement. Let $(\kappa_p)_p$ be an analytic cut-off sequence for $\overline{\Omega'}$ supported in Ω . Lemma 6.2.1 implies that there are $C, h > 0$ such that

$$|\xi|^p |\widehat{\kappa_p \varphi}(\xi)| \leq Ch^p M_p, \quad p \in \mathbb{N}, \xi \in \mathbb{R}^d.$$

Set $p_n = m(n/(Hh)) + d + 1$ for $n \in \mathbb{N}$. By the first part of the proof it suffices to show that

$$((\kappa_{p_n} \varphi) * \rho_n - \varphi)_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega').$$

Again, we use Lemma 3.4.5 to this end (notice that the sequence $((\kappa_{p_n} \varphi) * \rho_n - \varphi)_n$ is moderate by (6.4.2)). For $n \geq 1$ it holds that

$$\begin{aligned} & \sup_{x \in \Omega'} |((\kappa_{p_n} \varphi) * \rho_n)(x) - \varphi(x)| \\ & \leq \sup_{x \in \Omega'} |((\kappa_{p_n} \varphi) * \rho_n)(x) - (\kappa_{p_n} \varphi)(x)| \\ & \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{\kappa_{p_n} \varphi}(\xi)| (1 - \chi_n(\xi/n)) d\xi \\ & \leq \frac{C_0 C (Hh)^{d+1} M_{d+1}}{(2\pi)^d} \int_{|\xi| \geq n} \frac{(Hh)^{m(n/(Hh))} M_{m(n/(Hh))}}{|\xi|^{m(n/(Hh))+d+1}} d\xi \\ & \leq C' e^{-M(n/(Hh))}, \end{aligned}$$

where

$$C' = \frac{C_0 C (Hh)^{d+1} M_{d+1}}{(2\pi)^d} \int_{|\xi| \geq 1} \frac{1}{|\xi|^{d+1}} d\xi < \infty,$$

□

Proof of Proposition 6.4.2. The assertion concerning the ultradifferential operators of class $\{M_p\}$ is clear from the definition of ι_c . The rest of the proof is divided into three steps.

STEP I: $(f * \rho_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\mathbb{R}^d)$ for all $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$. Let $\lambda > 0$ be arbitrary. Choose $C > 0$ such that $|\widehat{f}(\xi)| \leq C e^{M(\lambda\xi/(Hr))}$, $\xi \in \mathbb{R}^d$. We have that

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{F}(f * \rho_n)(\xi)| e^{M(\lambda\xi/(H^2r))} d\xi &= \int_{\mathbb{R}^d} |\widehat{f}(\xi)| \chi_n(\xi/n) e^{M(\lambda\xi/(H^2r))} d\xi \\ &\leq C' e^{M(\lambda n)}, \end{aligned}$$

where

$$C' = C_0^2 C \int_{\mathbb{R}^d} e^{-M(\lambda\xi/(H^2 r))} d\xi < \infty.$$

The result now follows from Lemma 2.2.7.

STEP II: $\text{supp } \iota_c(f) \subseteq \text{supp } f$ for all $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$. Let $K \Subset \mathbb{R}^d \setminus \text{supp } f$ be arbitrary. Choose $K' \Subset \mathbb{R}^d$ such that $\text{supp } f \subset \text{int } K'$ and $K' \cap K = \emptyset$. Set $d(K, K') = c > 0$ and choose $C, h, \lambda > 0$ as in (6.4.1). Since f is continuous, there is $C' > 0$ such that

$$\begin{aligned} \max_{x \in K} |(f * \rho_n)(x)| &\leq C' \|\rho_n\|_{K-K', h} \\ &\leq C' \sup_{\alpha \in \mathbb{N}^d} \max_{x \in K-K'} \frac{|\partial^\alpha \rho_n(x)|}{h^{|\alpha|} M_\alpha} \leq CC' e^{-M(\lambda n)}, \end{aligned}$$

whence $\iota_c(f)$ vanishes in $\mathbb{R}^d \setminus \text{supp } f$ because of Lemma 3.4.5.

STEP III: $\text{supp } f \subseteq \text{supp } \iota_c(f)$ for all $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$. Set $K = \text{supp } \iota_c(f)$. We need to show that $f \in \mathcal{E}'^{\{M_p\}}(\Omega)$ for every open set Ω in \mathbb{R}^d such that $K \Subset \Omega$. By Proposition 5.2.4 there is a weight sequence N_p satisfying (M.1) and $M_p \prec N_p$ such that $f \in \mathcal{E}'^{\{N_p\}}(\mathbb{R}^d)$. Moreover, we may assume that N_p satisfies (M.2) by Lemma 2.2.3. Choose $\theta \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \leq \theta \leq 1$ and $\theta \equiv 1$ in a neighbourhood of $\text{supp } f$. Proposition 6.4.3 yields that

$$\langle f, \varphi \rangle = \lim_{n \rightarrow \infty} \langle f, (\theta \varphi) * \rho_n \rangle = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} (f * \rho_n)(x) \theta(x) \varphi(x) dx$$

for all $\varphi \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$. Next, choose $\kappa \in \mathcal{D}(\Omega)$ such that $0 \leq \kappa \leq 1$ and $\kappa \equiv 1$ in a neighbourhood of K . By STEP II we have that $\kappa - \theta \equiv 0$ in a neighbourhood of K . In particular, $\iota_c(f)|_{\mathbb{R}^d \setminus K} = 0$ implies that $f * \rho_n \rightarrow 0$ uniformly on $\text{supp}(\kappa - \theta)$. Hence,

$$\langle f, \varphi \rangle = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} (f * \rho_n)(x) \kappa(x) \varphi(x) dx, \quad \varphi \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d). \quad (6.4.3)$$

We now invoke Lemma 6.2.2. Set $\psi_0 = \chi_1$ and

$$\psi_n = \chi_{n+1} \left(\frac{\cdot}{n+1} \right) - \chi_n \left(\frac{\cdot}{n} \right), \quad n \geq 1.$$

Clearly, the sequence $(\psi_n)_n$ satisfies the requirements of Lemma 6.2.2. Choose a relatively compact open subset Ω' in Ω such that $K \subseteq \Omega'$ and an analytic cut-off sequence $(\kappa_p)_p$ for K supported in Ω' . According to Lemma 6.2.2 (applied to the weight sequence N_p), there are a sequence $(p_n)_n$ and $k > 0$ such that the mapping $R : \mathcal{E}^{N_p,1}(\overline{\Omega'}) \rightarrow \mathcal{B}^{N_p,k}(\mathbb{R}^d)$ given by

$$R(\varphi) = \sum_{n=0}^{\infty} (\kappa_{p_n} \varphi) * \mathcal{F}^{-1}(\psi_n)$$

is well-defined and continuous. Consider the inclusion mappings $\iota_1 : \mathcal{E}^{\{M_p\}}(\Omega) \rightarrow \mathcal{E}^{N_p,1}(\overline{\Omega'})$ and $\iota_2 : \mathcal{B}^{N_p,k}(\mathbb{R}^d) \rightarrow \mathcal{E}^{\{N_p\}}(\mathbb{R}^d)$. We set $T = \iota_2 \circ R \circ \iota_1 : \mathcal{E}^{\{M_p\}}(\Omega) \rightarrow \mathcal{E}^{\{N_p\}}(\mathbb{R}^d)$. Equality (6.4.3) implies that

$$\begin{aligned} \langle f, \varphi \rangle &= \int_{\mathbb{R}^d} \sum_{n=0}^{\infty} (f * \mathcal{F}^{-1}(\psi_n))(x) (\kappa(x) - \kappa_{p_n}(x)) \varphi(x) dx \\ &\quad + \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} (f * \mathcal{F}^{-1}(\psi_n))(x) \kappa_{p_n}(x) \varphi(x) dx \end{aligned}$$

for all $\varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$. Since $(\kappa - \kappa_{p_n})_n$ is a bounded sequence in $\mathcal{D}(\Omega \setminus K)$, the assumption $\iota_c(f)|_{\mathbb{R}^d \setminus K} = 0$ yields that

$$g = \sum_{n=0}^{\infty} (f * \mathcal{F}^{-1}(\psi_n))(\kappa - \kappa_{p_n}) \in \mathcal{D}(\Omega).$$

For the second term, we have that

$$\begin{aligned} \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} (f * \mathcal{F}^{-1}(\psi_n))(x) \kappa_{p_n}(x) \varphi(x) dx &= \sum_{n=0}^{\infty} \langle f, \mathcal{F}^{-1}(\psi_n) * (\kappa_{p_n} \varphi) \rangle \\ &= \langle f, T(\varphi) \rangle \end{aligned}$$

for all $\varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$. Hence, $f = g + {}^tT(f) \in \mathcal{E}'^{\{M_p\}}(\Omega)$. \square

We have completed all necessary work to prove the main theorem of this chapter.

Theorem 6.4.4. *Let M_p be a weight sequence satisfying (M.1), (M.2), (M.2)*, (QA), and (NE). Then, there is a unique sheaf monomorphism $\iota : \mathcal{B}^{\{M_p\}} \rightarrow \mathcal{G}^{\{M_p\}}$ such that:*

- (i) *The restriction of $\iota_{\mathbb{R}^d}$ to $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$ coincides with ι_c .*
- (ii) *The restriction of ι to $\mathcal{E}^{\{M_p\}}$ coincides with σ . In particular, $\iota_\Omega(\varphi\psi) = \iota_\Omega(\varphi) \cdot \iota_\Omega(\psi)$ for all open subsets Ω of \mathbb{R}^d and all $\varphi, \psi \in \mathcal{E}^{\{M_p\}}(\Omega)$.*
- (iii) *For each ultradifferential operator $P(D)$ of class $\{M_p\}$ we have that $\tilde{P}(D) \circ \iota = \iota \circ P(D)$.*

Proof. Since $\mathcal{B}^{\{M_p\}}$ and $\mathcal{G}^{\{M_p\}}$ are soft sheaves (Propositions 2.2.11 and 6.3.2), the existence and uniqueness of a sheaf embedding ι satisfying the first and third property directly follows from Lemma 2.2.15 (cf. Section 3.3). We now show that $\iota_\Omega(\varphi) = \sigma_\Omega(\varphi)$ for all open subsets Ω of \mathbb{R}^d and all $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega)$. It suffices to show that $\iota_\Omega(\varphi)|_{\Omega'} = \sigma_\Omega(\varphi)|_{\Omega'}$ for all relatively compact open subsets Ω' of Ω . Choose $\kappa \in \mathcal{D}(\Omega)$ such that $0 \leq \kappa \leq 1$ and $\kappa \equiv 1$ in a neighbourhood of $\overline{\Omega'}$. Notice that $\iota_\Omega(\varphi)|_{\Omega'} = \iota_{\Omega'}(\varphi|_{\Omega'}) = \iota_{\Omega'}((\kappa\varphi)|_{\Omega'}) = \iota_\Omega(\kappa\varphi)|_{\Omega'} = \iota_c(\kappa\varphi)|_{\Omega'}$. Hence, we only need to show that $((\kappa\varphi) * \rho_n - \varphi)_n \in \mathcal{E}_N^{\{M_p\}}(\Omega')$, but this has already been proved in Proposition 6.4.3. \square

Chapter 7

Diffeomorphism invariant Colombeau algebras

7.1 Introduction

Perhaps the most appealing feature of the special Colombeau algebra is its simple structure but, as pointed out in Chapter 3, there is no canonical embedding of distributions. A related and more severe problem is the fact that there is no induced action of diffeomorphisms extending the classical pullback of distributions (cf. [70, Sect. 3.2.2]), which is a desirable property from a geometrical point of view. Alongside the special Colombeau algebras there is also the *full* variant of Colombeau algebras [29, 70]. These are technically more involved but they do not suffer the aforementioned drawbacks. Over the past two decades, such algebras were employed to develop an intrinsic geometric non-linear theory of generalized functions [69, 71, 70, 121]. Full diffeomorphism invariant Colombeau algebras, defined on open subsets of \mathbb{R}^d , were for the first time obtained in [69]. This work served as the basis for the construction of full diffeomorphism invariant Colombeau algebras defined on general manifolds in [71]. These algebras are rather hard to grasp due to the heavy technical apparatus needed to define

them. Recently, Nigsch proposed a more conceptual approach to Colombeau theory [120, 119] based on classical functional analytic notions. This approach allows for a great flexibility and clarifies the relationship between the various Colombeau algebras defined in the literature; in this regard, see also [72]. For example, it can be used to construct diffeomorphism invariant Colombeau algebras on manifolds in a very efficient and transparent way.

In this chapter, we shall further develop the functional analytic approach to Colombeau algebras from [120, 119]. Our main goal is to formulate the theory in a more abstract way in order to increase the scope of possible applications. In particular, our results provide a unifying framework for diffeomorphism invariant Colombeau algebras containing spaces of distributions as well as ultradistributions, both of Beurling and Roumieu type. As in Chapter 3, the basic idea is to replace the pair of sheaves (\mathcal{D}', C^∞) by a rather general pair of sheaves (E, F) satisfying certain natural compatibility conditions.

This chapter is based on joint work with E.A. Nigsch [37] but, in fact, all the conceptual ideas were already contained in his papers [120, 119]. I would like to thank Eduard for giving me the opportunity to work together with him on this fascinating subject.

This chapter is organized as follows. In Section 7.2, we introduce the basic spaces containing the representatives of non-linear generalized functions. The quotient construction, which ensures that the product of sufficiently regular functions is preserved, is detailed in Section 7.3. In Section 7.4, we thoroughly discuss the sheaf properties of the quotient spaces. Finally, as an application of our general theory, we construct diffeomorphism invariant Colombeau algebras containing spaces of ultradistributions in Section 7.5.

7.2 The basic space

In this section we set up the general framework and define the basic spaces. Let X and Y be locally convex spaces. We denote by $C^\infty(X, Y)$ the space of smooth functions from X into Y in the sense of convenient calculus [101]; in this context, $d^k f$ denotes the k -th differential of a mapping $f \in C^\infty(X, Y)$. We refer to [101] for more information on calculus in infinite dimensional spaces.

A pair (E, F) of l.c.s. is called a *test pair* if $F \subseteq E$ and the topology on F is finer than the one induced by E . Throughout this section we fix a test pair (E, F) .

We define the *basic space* as

$$\mathcal{E}(E, F) := C^\infty(L_b(E, F), F)$$

and the canonical linear embeddings of E and F into $\mathcal{E}(E, F)$ via

$$\iota : E \rightarrow \mathcal{E}(E, F) : f \rightarrow (\Phi \rightarrow \Phi(f))$$

and

$$\sigma : F \rightarrow \mathcal{E}(E, F) : \varphi \rightarrow (\Phi \rightarrow \varphi),$$

respectively. There are three common ways of transferring operations T on E and F to the basic space $\mathcal{E}(E, F)$. Roughly speaking, these are given as follows:

$$\tilde{T}(R)(\Phi) := T(R(\Phi)).$$

$$\bar{T}(R)(\Phi) := T(R(T^{-1} \circ \Phi \circ T)).$$

$$\hat{T}(R)(\Phi) := T(R(\Phi)) - dR(\Phi)(T \circ \Phi - \Phi \circ T).$$

We will now specify in which situation they are well-defined on the basic space and when each variant is employed. The first one amounts to applying an operation on F after inserting the parameter $\Phi \in L(E, F)$. This defines the vector space structure of $\mathcal{E}(E, F)$ and its algebra structure if F is a locally convex algebra. Moreover, this is used for extending directional derivatives and especially the

covariant derivative in geometry (see [121]). For multilinear mappings it is formulated as follows.

Lemma 7.2.1. *Let $T : F \times \cdots \times F \rightarrow F$ be a jointly continuous multilinear mapping. Then, the mapping $\tilde{T} : \mathcal{E}(E, F) \times \cdots \times \mathcal{E}(E, F) \rightarrow \mathcal{E}(E, F)$ given by*

$$\tilde{T}(R_1, \dots, R_k)(\Phi) := T(R_1(\Phi), \dots, R_k(\Phi)) \quad (7.2.1)$$

commutes with the embedding σ in the sense that

$$\tilde{T}(\sigma(\varphi_1), \dots, \sigma(\varphi_k)) = \sigma(T(\varphi_1, \dots, \varphi_k)).$$

Corollary 7.2.2. *Suppose that F is a locally convex algebra. Then, $\mathcal{E}(E, F)$ is an algebra with multiplication given by*

$$(R_1 \cdot R_2)(\Phi) := R_1(\Phi) \cdot R_2(\Phi) \quad (7.2.2)$$

and σ is an algebra homomorphism.

The second variant of extending operations to the basic space applies to isomorphisms on E which restrict to isomorphisms on F . This will be used for isomorphisms on ultradistribution spaces coming from diffeomorphisms of the respective domains.

Lemma 7.2.3. *Let (E_1, F_1) and (E_2, F_2) be two test pairs. Suppose that $T : E_1 \rightarrow E_2$ is a linear topological isomorphism such that also the restriction $T|_{F_1}$ is a linear topological isomorphism $F_1 \rightarrow F_2$. Then, the mapping $\bar{T} : \mathcal{E}(E_1, F_1) \rightarrow \mathcal{E}(E_2, F_2)$ given by*

$$\bar{T}(R)(\Phi) := T(R(T^{-1} \circ \Phi \circ T)) \quad (7.2.3)$$

is an isomorphism that makes the following diagrams commutative:

$$\begin{array}{ccc} E_1 & \xrightarrow{T} & E_2 \\ \iota \downarrow & & \downarrow \iota \\ \mathcal{E}(E_1, F_1) & \xrightarrow{\bar{T}} & \mathcal{E}(E_2, F_2) \end{array} \quad \begin{array}{ccc} F_1 & \xrightarrow{T} & F_2 \\ \sigma \downarrow & & \downarrow \sigma \\ \mathcal{E}(E_1, F_1) & \xrightarrow{\bar{T}} & \mathcal{E}(E_2, F_2) \end{array}$$

Finally, the third variant of extending operations to the basic space applies to the extension of derivatives to $\mathcal{E}(E, F)$. In the following lemma, the notation RO stands for “regularization operator”.

Lemma 7.2.4. *Let $T \in L(E, E)$ with $T|_F \in L(F, F)$. Then, the mapping*

$$T^{\text{RO}} : L_b(E, F) \rightarrow L_b(E, F) : \Phi \rightarrow T \circ \Phi - \Phi \circ T$$

is linear and continuous, and the mapping $\widehat{T} : \mathcal{E}(E, F) \rightarrow \mathcal{E}(E, F)$ given by

$$\widehat{T}(R)(\Phi) := T(R(\Phi)) - \text{d}R(\Phi)(T^{\text{RO}}\Phi) \quad (7.2.4)$$

commutes with the embeddings ι and σ in the sense that $\widehat{T} \circ \iota = \iota \circ T$ and $\widehat{T} \circ \sigma = \sigma \circ T$.

7.3 The quotient construction

Colombeau algebras are defined as the quotient of moderate by negligible functions, which ensures that the product of sufficiently regular functions is preserved. While originally these properties were determined by inserting translated and scaled test functions into the representatives of generalized functions, the functional analytic approach makes it possible to give an elegant formulation of this testing procedure in more general terms. Our next goal is to define moderateness and negligibility of elements of the basic space in the present setting. We start by introducing the scales of weights to be employed in the sequel. A set \mathcal{A} consisting of sequences of positive real numbers is said to be an *asymptotic growth scale* if the following conditions hold:

- $\forall \lambda, \mu \in \mathcal{A} \exists \nu \in \mathcal{A} : \sup_{n \in \mathbb{N}} (\lambda_n + \mu_n) / \nu_n < \infty.$
- $\forall \lambda, \mu \in \mathcal{A} \exists \nu \in \mathcal{A} : \sup_{n \in \mathbb{N}} \lambda_n \mu_n / \nu_n < \infty.$

- $\exists \lambda \in \mathcal{A} : \liminf_{n \rightarrow \infty} \lambda_n > 0$.

Similarly, a set \mathcal{I} consisting of sequences of positive real numbers is said to be an *asymptotic decay scale* if the following conditions hold:

- $\forall \lambda \in \mathcal{I} \exists \mu, \nu \in \mathcal{I} : \sup_{n \in \mathbb{N}} (\mu_n + \nu_n) / \lambda_n < \infty$.
- $\forall \lambda \in \mathcal{I} \exists \mu, \nu \in \mathcal{I} : \sup_{n \in \mathbb{N}} \mu_n \nu_n / \lambda_n < \infty$.
- $\exists \lambda \in \mathcal{I} : \lim_{n \rightarrow \infty} \lambda_n = 0$.

We call a pair $(\mathcal{A}, \mathcal{I})$ an *admissible pair of scales* if \mathcal{A} is an asymptotic growth scale, \mathcal{I} is an asymptotic decay scale, and the following two properties are satisfied:

- $\forall \lambda \in \mathcal{I} \forall \mu \in \mathcal{A} \exists \nu \in \mathcal{I} : \sup_{n \in \mathbb{N}} \mu_n \nu_n / \lambda_n < \infty$.
- $\exists \lambda \in \mathcal{A} \exists \mu \in \mathcal{I} : \sup_{n \in \mathbb{N}} \mu_n / \lambda_n < \infty$.

In this setting, the scale that is used in classical Colombeau theory is given by the polynomial scale

$$\mathcal{A} = \mathcal{I} = \{ (n^k)_n \mid k \in \mathbb{Z} \}.$$

Let (E, F) be a test pair and let $\text{sc} = (\mathcal{A}, \mathcal{I})$ be an admissible pair of scales. We define $\text{TO}(E, F, \text{sc})$ as the set consisting of all $(\Phi_n)_n \in L(E, F)^\mathbb{N}$ such that:

$$(\text{TO})_1 \quad \forall q \in \text{csn}(L_\sigma(E, F)) \exists \lambda \in \mathcal{A} : \sup_{n \in \mathbb{N}} q(\Phi_n) / \lambda_n < \infty.$$

$$(\text{TO})_2 \quad \forall q \in \text{csn}(L_\sigma(F, F)) \forall \lambda \in \mathcal{I} : \sup_{n \in \mathbb{N}} q(\Phi_n|_F - \text{id}_F) / \lambda_n < \infty.$$

$$(\text{TO})_3 \quad \Phi_n \rightarrow \text{id}_E \text{ in } L_\sigma(E, E).$$

Elements of $\text{TO}(E, F, \text{sc})$ are called *test objects (with respect to sc)*. If sc is clear from the context, we shall simply write $\text{TO}(E, F, \text{sc}) = \text{TO}(E, F)$. Similarly, we define $\text{TO}^0(E, F) = \text{TO}^0(E, F, \text{sc})$ as the set consisting of all $(\Psi_n)_n \in L(E, F)^\mathbb{N}$ satisfying

$$(\text{TO})_1^0 \quad \forall q \in \text{csn}(L_\sigma(E, F)) \exists \lambda \in \mathcal{A} : \sup_{n \in \mathbb{N}} q(\Psi_n)/\lambda_n < \infty.$$

$$(\text{TO})_2^0 \quad \forall q \in \text{csn}(L_\sigma(F, F)) \forall \lambda \in \mathcal{I} : \sup_{n \in \mathbb{N}} q(\Psi_n|_F)/\lambda_n < \infty.$$

$$(\text{TO})_3^0 \quad \Psi_n \rightarrow 0 \text{ in } L_\sigma(E, E).$$

Elements of $\text{TO}^0(E, F, \text{sc})$ are called *0-test objects (with respect to sc)*. We shall need the following result later on.

Lemma 7.3.1.

- (i) Let $T_i \in L(E, E)$, $i = 0, \dots, k$, $k \in \mathbb{N}$, be given such that $T_{i|F} \in L(F, F)$ and $\sum_{i=0}^k T_i = \text{id}$. Then, $\left(\sum_{i=0}^k T_i \circ \Phi_{i,n}\right)_n \in \text{TO}(E, F)$ for all $(\Phi_{i,n})_n \in \text{TO}(E, F)$, $i = 0, \dots, k$.
- (ii) Let $T \in L(E, E)$ be such that $T|_F \in L(F, F)$. Then, $(T \circ \Phi_n)_n \in \text{TO}^0(E, F)$ for all $(\Phi_n)_n \in \text{TO}^0(E, F)$.
- (iii) Let $T \in L(E, E)$ be such that $T|_F \in L(F, F)$. Then, $(T \circ \Phi_n - \Phi_n \circ T)_n \in \text{TO}^0(E, F)$ for all $(\Phi_n)_n \in \text{TO}(E, F) \cup \text{TO}^0(E, F)$.

We are now able to define moderateness and negligibility. Let $\text{sc} = (\mathcal{A}, \mathcal{I})$ be an admissible pair of scales and let $\Lambda \subseteq \text{TO}(E, F, \text{sc})$, $\Lambda^0 \subseteq \text{TO}^0(E, F, \text{sc})$ be non-empty. An element $R \in \mathcal{E}(E, F)$ is called *moderate (with respect to Λ , Λ^0 , and sc)* if

$$\begin{aligned} & \forall q \in \text{csn}(F) \quad \forall k \in \mathbb{N} \quad \forall (\Phi_n)_n \in \Lambda \\ & \quad \forall (\Psi_{1,n})_n, \dots, (\Psi_{k,n})_n \in \Lambda^0 \quad \exists \lambda \in \mathcal{A} : \\ & \quad \sup_{n \in \mathbb{N}} q(\text{d}^k R(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n}))/\lambda_n < \infty \end{aligned}$$

and *negligible (with respect to Λ , Λ^0 , and sc)* if

$$\begin{aligned} & \forall q \in \text{csn}(F) \quad \forall k \in \mathbb{N} \quad \forall (\Phi_n)_n \in \Lambda \\ & \quad \forall (\Psi_{1,n})_n, \dots, (\Psi_{k,n})_n \in \Lambda^0 \quad \forall \lambda \in \mathcal{I} : \\ & \quad \sup_{n \in \mathbb{N}} q(\text{d}^k R(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n}))/\lambda_n < \infty. \end{aligned}$$

The sets of all moderate and negligible elements are denoted by

$$\mathcal{E}_{\mathcal{M}}(E, F) = \mathcal{E}_{\mathcal{M}}(E, F, \Lambda, \Lambda^0, \text{sc})$$

and

$$\mathcal{E}_{\mathcal{N}}(E, F) = \mathcal{E}_{\mathcal{N}}(E, F, \Lambda, \Lambda^0, \text{sc}),$$

respectively.

The following important properties follow immediately from the definitions. In fact, the definitions were chosen in such a way precisely for these properties to hold.

Proposition 7.3.2.

- (i) $\mathcal{E}_{\mathcal{M}}(E, F)$ is a vector space and $\mathcal{E}_{\mathcal{N}}(E, F)$ is a subspace of $\mathcal{E}_{\mathcal{M}}(E, F)$.
- (ii) $\iota(E) \subseteq \mathcal{E}_{\mathcal{M}}(E, F)$ and $\sigma(F) \subseteq \mathcal{E}_{\mathcal{M}}(E, F)$.
- (iii) $\iota(E) \cap \mathcal{E}_{\mathcal{N}}(E, F) = \{0\}$ and $\sigma(F) \cap \mathcal{E}_{\mathcal{N}}(E, F) = \{0\}$.
- (iv) $(\iota - \sigma)(F) \subseteq \mathcal{E}_{\mathcal{N}}(E, F)$.

We now construct the quotient. Let $\text{sc} = (\mathcal{A}, \mathcal{I})$ be an admissible pair of scales and let $\Lambda \subseteq \text{TO}(E, F, \text{sc})$, $\Lambda^0 \subseteq \text{TO}^0(E, F, \text{sc})$ be non-empty. The *non-linear extension of the test pair* (E, F) (with respect to Λ , Λ^0 , and sc) is defined as

$$\begin{aligned} \mathcal{G}(E, F) &= \mathcal{G}(E, F, \Lambda, \Lambda^0, \text{sc}) := \\ &\mathcal{E}_{\mathcal{M}}(E, F, \Lambda, \Lambda^0, \text{sc}) / \mathcal{E}_{\mathcal{N}}(E, F, \Lambda, \Lambda^0, \text{sc}). \end{aligned}$$

The equivalence class of $R \in \mathcal{E}_{\mathcal{M}}(E, F)$ is denoted by $[R]$. Proposition 7.3.2 implies that

$$\iota : E \rightarrow \mathcal{G}(E, F) : f \rightarrow [\iota(f)]$$

and

$$\sigma : F \rightarrow \mathcal{G}(E, F) : \varphi \rightarrow [\sigma(\varphi)]$$

are linear embeddings such that $\iota|_F = \sigma$. The name “non-linear extension” is justified by the following lemma.

Lemma 7.3.3. *Let $T : F \times \cdots \times F \rightarrow F$ be a jointly continuous multilinear mapping and consider the multilinear mapping $\tilde{T} : \mathcal{E}(E, F) \times \cdots \times \mathcal{E}(E, F) \rightarrow \mathcal{E}(E, F)$ given by (7.2.1). Then, \tilde{T} preserves moderateness, i.e.*

$$\tilde{T}(\mathcal{E}_{\mathcal{M}}(E, F), \dots, \mathcal{E}_{\mathcal{M}}(E, F)) \subseteq \mathcal{E}_{\mathcal{M}}(E, F),$$

and $\tilde{T}(R_1, \dots, R_k)$ is negligible if at least one of the R_i is negligible. Consequently, the mapping $\tilde{T} : \mathcal{G}(E, F) \times \cdots \times \mathcal{G}(E, F) \rightarrow \mathcal{G}(E, F)$ given by

$$\tilde{T}([R_k], \dots, [R_k]) := [\tilde{T}(R_1, \dots, R_k)]$$

is well-defined and satisfies

$$\tilde{T}(\sigma(\varphi_1), \dots, \sigma(\varphi_k)) = \sigma(T(\varphi_1, \dots, \varphi_k)).$$

Proof. This follows from Lemma 7.2.1 and the continuity of T . \square

Corollary 7.3.4. *Suppose that F is a locally convex algebra. Then, $\mathcal{E}_{\mathcal{M}}(E, F)$ is an algebra with multiplication given by (7.2.2) and $\mathcal{E}_{\mathcal{N}}(E, F)$ is an ideal of $\mathcal{E}_{\mathcal{M}}(E, F)$. Consequently, $\mathcal{G}(E, F)$ is an algebra with multiplication given by*

$$[R_1] \cdot [R_2] := [R_1 \cdot R_2]$$

and σ is an algebra homomorphism.

Lemma 7.3.5. *Let (E_1, F_1) and (E_2, F_2) be two test pairs. Suppose that $T : E_1 \rightarrow E_2$ is a linear topological isomorphism such that also the restriction $T|_{F_1}$ is a linear topological isomorphism $F_1 \rightarrow F_2$. Let $\text{sc} = (\mathcal{A}, \mathcal{I})$ be an admissible pair of scales and let $\Lambda_i \subseteq \text{TO}(E_i, F_i, \text{sc})$, $\Lambda_i^0 \subseteq \text{TO}^0(E_i, F_i, \text{sc})$ be non-empty, $i = 1, 2$, such that*

$$\begin{aligned} (T^{-1} \circ \Phi_n \circ T)_n &\in \Lambda_1, & (\Phi_n)_n &\in \Lambda_2, \\ (T^{-1} \circ \Psi_n \circ T)_n &\in \Lambda_1^0, & (\Psi_n)_n &\in \Lambda_2^0, \\ (T \circ \Phi_n \circ T^{-1})_n &\in \Lambda_2, & (\Phi_n)_n &\in \Lambda_1, \\ (T \circ \Psi_n \circ T^{-1})_n &\in \Lambda_2^0, & (\Psi_n)_n &\in \Lambda_1^0. \end{aligned}$$

Consider the mapping $\bar{T} : \mathcal{E}(E_1, F_1) \rightarrow \mathcal{E}(E_2, F_2)$ given by (7.2.3).
Set

$$\mathcal{E}_{\mathcal{M}}(E_i, F_i) = \mathcal{E}_{\mathcal{M}}(E_i, F_i, \Lambda_i, \Lambda_i^0, \text{sc})$$

and

$$\mathcal{E}_{\mathcal{N}}(E_i, F_i) = \mathcal{E}_{\mathcal{N}}(E_i, F_i, \Lambda_i, \Lambda_i^0, \text{sc})$$

for $i = 1, 2$. Then, \bar{T} preserves moderateness and negligibility.
Consequently, the mapping $\bar{T} : \mathcal{G}(E_1, F_1) \rightarrow \mathcal{G}(E_2, F_2)$ given by

$$\bar{T}([R]) := [\bar{T}(R)]$$

is a well-defined isomorphism that makes the following diagram commutative:

$$\begin{array}{ccc} E_1 & \xrightarrow{T} & E_2 \\ \downarrow \iota & & \downarrow \iota \\ \mathcal{G}(E_1, F_1) & \xrightarrow{\bar{T}} & \mathcal{G}(E_2, F_2) \end{array}$$

Proof. This follows from Lemma 7.2.3 and the continuity of T . \square

Lemma 7.3.6. Let $T \in L(E, E)$ with $T|_F \in L(F, F)$. Consider the mapping $\hat{T} : \mathcal{E}(E, F) \rightarrow \mathcal{E}(E, F)$ given by (7.2.4). Then, \hat{T} preserves moderateness and negligibility. Consequently, the mapping $\hat{T} : \mathcal{G}(E, F) \rightarrow \mathcal{G}(E, F)$ given by

$$\hat{T}([R]) := [\hat{T}(R)]$$

is well-defined and satisfies $\hat{T} \circ \iota = \iota \circ T$.

Proof. This follows from Lemma 7.2.4 and the continuity of T . \square

7.4 Sheaf properties

In this section, we study the sheaf theoretic properties of our quotient spaces. After introducing the necessary terminology, we first look in detail at test objects. Satisfying a certain localizability

condition, the spaces of test objects and 0-test objects themselves form sheaves. This property is used for showing the existence of global test objects by gluing together local ones and for extending and restricting test objects in the proof of the sheaf property of the Colombeau algebras.

7.4.1 Locally convex sheaves

Let X be a Hausdorff locally compact paracompact topological space. For open subsets V, U of X we write $V \subset\subset U$ to indicate that $\overline{V} \Subset U$; we shall only use this notation for *open* sets. A *locally convex sheaf* E on X is a sheaf E on X such that $E(U)$ is a locally convex space for each open subset $U \subseteq X$, the restriction mappings $\rho_{V,U}$ are continuous for every inclusion of open sets $V \subseteq U$, and for all $U \subseteq X$ open and all open coverings $(U_i)_i$ of U the following property is satisfied:

(S3) the topology on $E(U)$ coincides with the projective topology on $E(U)$ with respect to the mappings $\rho_{U_i, U}$.

Property (S3) and the fact that X is locally compact imply the ensuing isomorphism of l.c.s.

$$E(U) \cong \varprojlim_{W \subset\subset U} E(W), \quad (7.4.1)$$

Notice that the algebraic isomorphism in (7.4.1) holds because of (S1) and (S2).

A sheaf morphism $\mu : E_1 \rightarrow E_2$ between two locally convex sheaves on X is a collection of continuous linear mappings $\mu_U : E_1(U) \rightarrow E_2(U)$, $U \subseteq X$ open, such that the identity $\rho_{V,U} \circ \mu_U = \mu_V \circ \rho_{V,U}$ holds for every inclusion of open sets $V \subseteq U$. We denote by $\text{Hom}(E_1, E_2)$ the sheaf of all sheaf morphisms between E_1 and E_2 . A multilinear sheaf morphism $T : E \times \cdots \times E \rightarrow E$ is a collection of jointly continuous multilinear mappings $T_U : E(U) \times \cdots \times E(U) \rightarrow$

$E(U)$, $U \subseteq X$ open, such that, for every inclusion of open sets $V \subseteq U$, it holds that

$$\rho_{V,U}(T_U(f_1, \dots, f_k)) = T_V(\rho_{V,U}(f_1), \dots, \rho_{V,U}(f_k)).$$

A (locally convex) sheaf E is called a *(locally convex) sheaf of algebras* if $E(U)$ is a (locally convex) algebra for all $U \subseteq X$ open and the multiplication is a bilinear sheaf morphism.

7.4.2 Localizing regularization operators

Let E and F be locally convex sheaves. We call (E, F) a *test pair of sheaves* if the following three properties are satisfied:

- F is a subsheaf of E .
- $(E(U), F(U))$ is a test pair for each open set $U \subseteq X$.

Let $\mu \in \text{Hom}(E, E)$. We write $\mu|_F$ for its restriction to F . Hence, $\mu|_F \in \text{Hom}(F, F)$ means that $\mu_{U|F(U)}$ is a continuous linear operator on $F(U)$ for all $U \subseteq X$ open. The third property can then be formulated as follows:

- For all $U \subseteq X$ open and all closed subsets A, B of U with $A \cap B = \emptyset$ there is $\mu \in \text{Hom}(E|_U, E|_U)$ with $\mu|_F \in \text{Hom}(F|_U, F|_U)$ such that $\mu_V = \text{id}$ and $\mu_W = 0$ for some open neighbourhoods V and W (in U) of A and B , respectively. Or, equivalently, for every open set U in X and every open covering $(U_i)_i$ of U there is a partition of the unity $(\eta^i)_i \subset \text{Hom}(E|_U, E|_U)$ subordinate to $(U_i)_i$ such that $\eta^i_{|F|_U} \in \text{Hom}(F|_U, F|_U)$ for all i .

In particular, the third property yields that $E|_U$ and $F|_U$ are fine sheaves for all open sets $U \subseteq X$. Moreover, it implies that, for all open subsets U, V, W of X with $\overline{W} \subset V \subseteq U$, there exists $\tau \in L(E(V), E(U))$ with $\tau|_{F(V)} \in L(F(V), F(U))$ such that $\rho_{W,V} =$

$\rho_{W,U} \circ \tau$. Since F is a subsheaf of E , there is no need to make a distinction between the restriction mappings on E and F , respectively. These mappings will be denoted by $\rho_{U,V}$.

Let $U \subseteq X$ be open. We shall employ the short-hand notation $\text{RO}(U) = L(E(U), F(U))$, where RO stands for “regularization operator”. An element $(\Phi_n)_n \in \text{RO}(U)^{\mathbb{N}}$ is called *localizing* if

$$\forall V, V_0 \subseteq X : V \subset\subset V_0 \subset\subset U \exists n_0 \in \mathbb{N} \forall n \geq n_0 \forall f \in E(U) : \\ \rho_{V_0,U}(f) = 0 \Rightarrow \rho_{V,U}(\Phi_n(f)) = 0.$$

We write $\text{RO}_{\text{loc}}(U)$ for the set consisting of all localizing elements in $\text{RO}(U)^{\mathbb{N}}$. Furthermore, we define

$$\text{TO}_{\text{loc}}(U) = \text{TO}_{\text{loc}}(U, \text{sc}) := \text{TO}(E(U), F(U), \text{sc}) \cap \text{RO}_{\text{loc}}(U)$$

and

$$\text{TO}_{\text{loc}}^0(U) = \text{TO}_{\text{loc}}^0(U, \text{sc}) := \text{TO}^0(E(U), F(U), \text{sc}) \cap \text{RO}_{\text{loc}}(U),$$

where sc is an admissible pair of scales.

Remark 7.4.1. Throughout this section we shall always assume that the space $\text{TO}_{\text{loc}}(U)$ is non-empty.

We define $\text{NO}(U)$ as the vector space consisting of all $(\Phi_n)_n \in \text{RO}(U)^{\mathbb{N}}$ such that, for all $V \subset\subset U$, it holds that $\rho_{V,U} \circ \Phi_n = 0$ for n large enough. Define

$$\widetilde{\text{RO}}_{\text{loc}}(U) := \text{RO}_{\text{loc}}(U) / \text{NO}(U), \quad \widetilde{\text{TO}}_{\text{loc}}^0(U) := \text{TO}_{\text{loc}}^0(U) / \text{NO}(U).$$

For $(\Phi_n)_n, (\Phi'_n)_n \in \text{RO}(U)^{\mathbb{N}}$ we write $(\Phi_n)_n \sim (\Phi'_n)_n$ if $(\Phi_n - \Phi'_n)_n \in \text{NO}(U)$. Set

$$\widetilde{\text{TO}}_{\text{loc}}(U) := \text{TO}_{\text{loc}}(U) / \sim.$$

The main goal of this section is to show that one can define a natural sheaf structure on $\widetilde{\text{RO}}_{\text{loc}}$. We start by defining the restriction mappings.

Lemma 7.4.2. *Let U, V be open subsets of X such that $V \subseteq U$. Then, there is a continuous linear mapping $\rho_{V,U}^{\text{RO}} : \text{RO}(U) \rightarrow \text{RO}(V)$ such that, for all $(\Phi_n)_n \in \text{RO}_{\text{loc}}(U)$, the following properties hold:*

(i) *We have that*

$$\begin{aligned} & \forall W, W_0 \subseteq X : W \subset\subset W_0 \subset\subset V \\ & \exists n_0 \in \mathbb{N} \forall n \geq n_0 \forall f \in E(U) \forall g \in E(V) : \\ & \rho_{W_0,U}(f) = \rho_{W_0,V}(g) \Rightarrow \rho_{W,V}(\rho_{V,U}^{\text{RO}}(\Phi_n)(g)) = \rho_{W,U}(\Phi_n(f)). \end{aligned}$$

(ii) *For all $W \subset\subset V$ and all $\tau \in L(E(V), E(U))$ with $\rho_{W_0,U} \circ \tau = \rho_{W_0,V}$ for some $W \subset\subset W_0 \subset\subset V$ we have that*

$$\rho_{W,U} \circ \Phi_n \circ \tau = \rho_{W,V} \circ \rho_{V,U}^{\text{RO}}(\Phi_n)$$

for n large enough.

(iii) *For all $W \subset\subset V$ we have that*

$$\rho_{W,V} \circ \rho_{V,U}^{\text{RO}}(\Phi_n) \circ \rho_{V,U} = \rho_{W,U} \circ \Phi_n$$

for n large enough.

(iv) *For all $W \subset\subset V$ and $\Phi_1, \Phi_2 \in \text{RO}(U)$ with $\rho_{W,U} \circ \Phi_1 = \rho_{W,U} \circ \Phi_2$ we have that*

$$\rho_{W,V} \circ \rho_{V,U}^{\text{RO}}(\Phi_1) = \rho_{W,V} \circ \rho_{V,U}^{\text{RO}}(\Phi_2).$$

Proof. Let $(V_i)_i$ be an open covering of V such that $V_i \subset\subset V$ for all i . Let $(\eta^i)_i \subset \text{Hom}(F|_V, F|_V)$ be a partition of the unity subordinate to $(V_i)_i$ and choose $\tau_i \in L(E(V), E(U))$ such that $\rho_{V_i,V} = \rho_{V_i,U} \circ \tau_i$ for all i . We define

$$\rho_{V,U}^{\text{RO}}(\Phi) = \sum_i \eta_V^i \circ \rho_{V,U} \circ \Phi \circ \tau_i.$$

For all $W \subset\subset V$ it holds that $\text{supp } \eta^i \cap W = \emptyset$ except for i belonging to some finite index set I . Hence,

$$\rho_{W,V} \circ \rho_{V,U}^{\text{RO}}(\Phi) = \sum_{i \in I} \eta_W^i \circ \rho_{W,U} \circ \Phi \circ \tau_i. \quad (7.4.2)$$

By (7.4.1) we then have that $\rho_{V,U}^{\text{RO}}(\Phi) \in \text{RO}(V)$. The linearity and continuity of $\rho_{V,U}^{\text{RO}}$ and also (iv) are clear from this expression. We now show (i). Let $W \subset\subset V$ and $W \subset\subset W_0 \subset\subset V$ be arbitrary. Suppose that the representation (7.4.2) holds. Choose $V'_i \subset\subset V_i$ such that $\text{supp } \eta^i \subset V'_i$. Since $(\Phi_n)_n$ is localizing, there is $n_0 \in \mathbb{N}$ such that

$$\rho_{W_0 \cap V_i, U}(f) = 0 \Rightarrow \rho_{W \cap V'_i, U}(\Phi_n(f)) = 0 \quad (7.4.3)$$

for all $i \in I$, $n \geq n_0$, and $f \in E(U)$. Assume that $f \in E(U)$ and $g \in E(V)$ are given such that $\rho_{W_0, U}(f) = \rho_{W_0, V}(g)$. Since

$$\rho_{W, U} \circ \Phi_n = \sum_{i \in I} \eta_W^i \circ \rho_{W, U} \circ \Phi_n$$

and $\text{supp } \eta^i \subset V'_i$, it suffices to show that $\rho_{W \cap V'_i, U}(\Phi_n(f - \tau_i(g))) = 0$ for all $i \in I$, but this follows from (7.4.3). Properties (ii) and (iii) are special cases of (i). \square

Let U, V be open subsets of X such that $V \subseteq U$. In the sequel, we fix a continuous linear mapping $\rho_{V,U}^{\text{RO}} : \text{RO}(U) \rightarrow \text{RO}(V)$ satisfying the assumptions (i)-(iv) from Lemma 7.4.2.

Lemma 7.4.3. *Let U, V be open subsets of X such that $V \subseteq U$. Then, for all $(\Phi_n)_n \in \text{RO}_{\text{loc}}(U)$, it holds that:*

- (i) $(\rho_{V,U}^{\text{RO}}(\Phi_n))_n \in \text{RO}_{\text{loc}}(V)$.
- (ii) If $(\Phi_n)_n \sim 0$, then $(\rho_{V,U}^{\text{RO}}(\Phi_n))_n \sim 0$.
- (iii) $((\rho_{W,V}^{\text{RO}} \circ \rho_{V,U}^{\text{RO}})(\Phi_n))_n \sim (\rho_{W,U}^{\text{RO}}(\Phi_n))_n$ for $W \subseteq V \subseteq U$.

Proof. (i) Let $W \subset\subset V$ and $W \subset\subset W_0 \subset\subset V$ be arbitrary. Since $(\Phi_n)_n$ is localizing there is $n_1 \in \mathbb{N}$ such that

$$\rho_{W_0,U}(f) = 0 \Rightarrow \rho_{W,U}(\Phi_n(f)) = 0 \quad (7.4.4)$$

for all $n \geq n_1$ and all $f \in E(U)$. Choose $\tau \in L(E(V), E(U))$ such that $\rho_{W_0,U} \circ \tau = \rho_{W_0,V}$. By Lemma 7.4.2(ii) there is $n_2 \in \mathbb{N}$ such that $\rho_{W,V} \circ \rho_{V,U}^{\text{RO}}(\Phi_n) = \rho_{W,U} \circ \Phi_n \circ \tau$ for all $n \geq n_2$. Set $n_0 = \max\{n_1, n_2\}$. Let $g \in E(V)$ be such that $\rho_{W_0,V}(g) = 0$. Then,

$$\rho_{W,V}(\rho_{V,U}^{\text{RO}}(\Phi_n)(g)) = \rho_{W,U}(\Phi_n(\tau(g))) = 0$$

for all $n \geq n_0$.

(ii) Let $W \subset\subset V$ be arbitrary. Choose $\tau \in L(E(V), E(U))$ such that $\rho_{W_0,U} \circ \tau = \rho_{W_0,V}$. By Lemma 7.4.2(ii) we have that

$$\rho_{W,V} \circ \rho_{V,U}^{\text{RO}}(\Phi_n) = \rho_{W,U} \circ \Phi_n \circ \tau = 0$$

for n large enough because $(\Phi_n)_n \sim 0$.

(iii) Let $W_0 \subset\subset W$ be arbitrary. Fix an open set W'_0 such that $W_0 \subset\subset W'_0 \subset\subset W$. Choose $\tau \in L(E(V), E(U))$ such that $\rho_{W'_0,U} \circ \tau = \rho_{W'_0,V}$ and $\tau' \in L(E(W), E(V))$ such that $\rho_{W'_0,V} \circ \tau' = \rho_{W'_0,W}$. Hence, $\tau \circ \tau' \in L(E(W), E(U))$ and $\rho_{W'_0,U} \circ \tau \circ \tau' = \rho_{W'_0,W}$. By Lemma 7.4.2(ii) we have that

$$\begin{aligned} \rho_{W_0,W} \circ \rho_{W,V}^{\text{RO}}(\rho_{V,U}^{\text{RO}}(\Phi_n)) &= \rho_{W_0,V} \circ \rho_{V,U}^{\text{RO}}(\Phi_n) \circ \tau' \\ &= \rho_{W_0,U} \circ \Phi_n \circ \tau \circ \tau' = \rho_{W_0,W} \circ \rho_{W,U}^{\text{RO}}(\Phi_n) \end{aligned}$$

for n large enough. □

Lemma 7.4.3 implies that the mappings

$$\rho_{V,U}^{\text{RO}}([\Phi_n]_n) := [(\rho_{V,U}^{\text{RO}}(\Phi_n))_n]$$

define a presheaf structure on $\widetilde{\text{RO}}_{\text{loc}}$. We now show that it is in fact a sheaf.

Proposition 7.4.4. $\widetilde{\text{RO}}_{\text{loc}}$ is a sheaf.

Proof. Let $U \subseteq X$ be open and let $(U_i)_i$ be an open covering of U .

(S1) Suppose that $[(\Phi_n)_n] \in \widetilde{\text{RO}}_{\text{loc}}(U)$ such that

$$\rho_{U_i, U}^{\text{RO}}([(\Phi_n)_n]) = 0$$

for all i . We need to show that $(\Phi_n)_n \sim 0$. Let $W \subset\subset U$ be arbitrary. We may assume that $W \subset\subset U_i$ for some i . Lemma 7.4.2(iii) yields that

$$\rho_{W, U} \circ \Phi_n = \rho_{W, U_i} \circ \rho_{U_i, U}^{\text{RO}}(\Phi_n) \circ \rho_{U_i, U} = 0$$

for n large enough.

(S2) we may assume that $U_i \subset\subset U$ for all i . Suppose that $[(\Phi_{i,n})_n] \in \widetilde{\text{RO}}_{\text{loc}}(U_i)$ are given such that

$$\rho_{U_i \cap U_j, U_i}^{\text{RO}}([(\Phi_{i,n})_n]) = \rho_{U_i \cap U_j, U_j}^{\text{RO}}([(\Phi_{j,n})_n])$$

for all i, j . Let $(\eta^i)_i \subset \text{Hom}(F|_U, F|_U)$ be a partition of the unity subordinate to the covering $(U_i)_i$. Choose $\tau_i \in L(F(U_i), F(U))$ such that $\rho_{V_i, U} \circ \tau_i = \rho_{V_i, U_i}$ for some $V_i \subset\subset U_i$ with $\text{supp } \eta^i \subset V_i$. We define

$$\Phi_n = \sum_i \eta_U^i \circ \tau_i \circ \Phi_{i,n} \circ \rho_{U_i, U}$$

for all $n \in \mathbb{N}$. Notice that $\Phi_n \in \text{RO}(U)$ because of (7.4.1) and the fact that the family of supports of the η^i is locally finite. We now show that $(\Phi_n)_n$ is localizing. Let $W \subset\subset U$ and $W \subset\subset W_0 \subset\subset U$ be arbitrary and suppose that $\text{supp } \eta^i \cap W = \emptyset$ except for i belonging to some finite index set I . Choose $V'_i \subset\subset U_i$ such that $V_i \subset\subset V'_i$. Since the $(\Phi_{i,n})_n$ are localizing, there is $n_0 \in \mathbb{N}$ such that

$$\rho_{W_0 \cap V'_i, U_i}(f) = 0 \Rightarrow \rho_{W \cap V_i, U_i}(\Phi_{i,n}(f)) = 0 \quad (7.4.5)$$

for all $i \in I$, $n \geq n_0$, and $f \in E(U_i)$. Now suppose that $f \in E(U)$ satisfies $\rho_{W_0, U}(f) = 0$. Since

$$\rho_{W, U}(\Phi_n(f)) = \sum_{i \in I} \eta_W^i(\rho_{W, U}(\tau_i(\Phi_{i,n}(\rho_{U_i, U}(f)))))$$

and $\text{supp } \eta^i \subset V_i$, it suffices to show that

$$\rho_{W \cap V_i, U}(\tau_i(\Phi_{i,n}(\rho_{U_i, U}(f)))) = \rho_{W \cap V_i, U_i}(\Phi_{i,n}(\rho_{U_i, U}(f))) = 0$$

for all $i \in I$ and n large enough, but this follows from (7.4.5). Finally, we show that $\rho_{U_i, U}^{\text{RO}}([\Phi_n]) = [\Phi_{i,n}]$ for all i . Let $W \subset\subset U_i$ be arbitrary and suppose that $\text{supp } \eta^j \cap W = \emptyset$ except for j belonging to some finite index set I . Let $\tau \in L(E(U_i), E(U))$ be such that $\rho_{W_0, U} \circ \tau = \rho_{W_0, U_i}$, where W_0 is some open set such that $W \subset\subset W_0 \subset\subset U_i$. Lemma 7.4.2(ii) yields that

$$\begin{aligned} \rho_{W, U_i} \circ \rho_{U_i, U}^{\text{RO}}(\Phi_n) - \rho_{W, U_i} \circ \Phi_{i,n} &= \rho_{W, U} \circ \Phi_n \circ \tau - \rho_{W, U_i} \circ \Phi_{i,n} \\ &= \sum_{j \in I} \eta_W^j \circ (\rho_{W, U} \circ \tau_j \circ \Phi_{j,n} \circ \rho_{U_j, U} \circ \tau - \rho_{W, U_i} \circ \Phi_{i,n}). \end{aligned}$$

Since $\text{supp } \eta^j \subset V_j$ it suffices to show that

$$\rho_{W \cap V_j, U} \circ \tau_j \circ \Phi_{j,n} \circ \rho_{U_j, U} \circ \tau - \rho_{W \cap V_j, U_i} \circ \Phi_{i,n} = 0$$

for all $j \in I$ and n large enough. Lemma 7.4.2(ii) and (iii) imply that

$$\begin{aligned} &\rho_{W \cap V_j, U} \circ \tau_j \circ \Phi_{j,n} \circ \rho_{U_j, U} \circ \tau - \rho_{W \cap V_j, U_i} \circ \Phi_{i,n} \\ &= \rho_{W \cap V_j, U_j} \circ \Phi_{j,n} \circ \rho_{U_j, U} \circ \tau - \rho_{W \cap V_j, U_i} \circ \Phi_{i,n} \\ &= \rho_{W \cap V_j, U_i \cap U_j} \circ \rho_{U_i \cap U_j, U_j}^{\text{RO}}(\Phi_{j,n}) \circ \rho_{U_i \cap U_j, U} \circ \tau - \rho_{W \cap V_j, U_i} \circ \Phi_{i,n} \\ &= \rho_{W \cap V_j, U_i \cap U_j} \circ \rho_{U_i \cap U_j, U_i}^{\text{RO}}(\Phi_{i,n}) \circ \rho_{U_i \cap U_j, U} \circ \tau - \rho_{W \cap V_j, U_i} \circ \Phi_{i,n} \\ &= \rho_{W \cap V_j, U_i} \circ \Phi_{i,n} \circ \rho_{U_i, U} \circ \tau - \rho_{W \cap V_j, U_i} \circ \Phi_{i,n}, \end{aligned}$$

which equals zero for n large enough because $(\Phi_{i,n})_n$ is localizing. \square

Lemma 7.4.5. *Every sheaf morphism $\mu \in \text{Hom}(F, F)$ induces a sheaf morphism $\mu \in \text{Hom}(\widehat{\text{RO}}_{\text{loc}}, \widehat{\text{RO}}_{\text{loc}})$ via*

$$\mu_U([\Phi_n]) := [(\mu_U \circ \Phi_n)_n]. \quad (7.4.6)$$

Proof. Clearly, $\mu_U : \widetilde{\text{RO}}_{\text{loc}}(U) \rightarrow \widetilde{\text{RO}}_{\text{loc}}(U)$ is a well-defined linear mapping for all $U \subseteq X$ open. We now show that μ is a sheaf morphism. Let V, U be open subsets of X such that $V \subseteq U$. It suffices to show that for all $W \subset\subset V$ and all $(\Phi_n)_n \in \text{RO}_{\text{loc}}(U)$ it holds that $\rho_{W,V} \circ \rho_{V,U}^{\text{RO}}(\mu_U \circ \Phi_n) = \rho_{W,V} \circ \mu_V \circ \rho_{V,U}^{\text{RO}}(\Phi_n)$ for n large enough. Let $\tau \in L(E(V), E(U))$ be such that $\rho_{W_0,U} \circ \tau = \rho_{W_0,V}$ for some open set W_0 such that $W \subset\subset W_0 \subset\subset V$. By Lemma 7.4.2(ii) we have that

$$\begin{aligned} \rho_{W,V} \circ \rho_{V,U}^{\text{RO}}(\mu_U \circ \Phi_n) &= \rho_{W,U} \circ \mu_U \circ \Phi_n \circ \tau = \mu_W \circ \rho_{W,U} \circ \Phi_n \circ \tau \\ &= \mu_W \circ \rho_{W,V} \circ \rho_{V,U}^{\text{RO}}(\Phi_n) = \rho_{W,V} \circ \mu_V \circ \rho_{V,U}^{\text{RO}}(\Phi_n) \end{aligned}$$

for n large enough. \square

We now turn our attention to spaces of test objects.

Lemma 7.4.6. *Let $U \subseteq X$ be open and let $(U_i)_i$ be an open covering of U . Let $(\Phi_n)_n \in \text{RO}_{\text{loc}}(U)$. Then, $(\Phi_n)_n \in \text{TO}_{\text{loc}}(U)$ if and only if $(\rho_{U_i,U}^{\text{RO}}(\Phi_n))_n \in \text{TO}_{\text{loc}}(U_i)$ for all i . Moreover, a similar statement holds for TO_{loc}^0 .*

Proof. We only show the statement for TO_{loc} ; the proof for TO_{loc}^0 is similar. We first assume that $(\Phi_n)_n$ satisfies $(\text{TO})_j$, $j = 1, 2, 3$, and prove that $(\rho_{U_i,U}^{\text{RO}}(\Phi_n))_n$ does so as well.

$(\text{TO})_1$ It suffices to show that, for all $f \in E(U_i)$ and $q \in \text{csn}(F(W))$, with $W \subset\subset U_i$ arbitrary, there is $\lambda \in \mathcal{A}$ such that

$$\sup_{n \in \mathbb{N}} q(\rho_{W,U_i}(\rho_{U_i,U}^{\text{RO}}(\Phi_n)(f))) / \lambda_n < \infty.$$

Let $\tau \in L(E(U_i), E(U))$ be such that $\rho_{W_0,U} \circ \tau = \rho_{W_0,U_i}$, where W_0 is an open set such that $W \subset\subset W_0 \subset\subset U_i$. By Lemma 7.4.2(ii) we have that

$$\rho_{W,U_i}(\rho_{U_i,U}^{\text{RO}}(\Phi_n)(f)) = \rho_{W,U}(\Phi_n(\tau(f)))$$

for n large enough. The result now follows from the fact that $\rho_{W,U} \in L(F(U), F(W))$.

(TO)₂ It suffices to show that, for all $\varphi \in F(U_i)$, $q \in \text{csn}(F(W))$, with $W \subset\subset U_i$ arbitrary, and $\lambda \in \mathcal{I}$ it holds that

$$\sup_{n \in \mathbb{N}} q(\rho_{W,U_i}(\rho_{U_i,U}^{\text{RO}}(\Phi_n)(\varphi) - \varphi)) / \lambda_n < \infty.$$

Let $\tau \in L(F(U_i), F(U))$ be such that $\rho_{W_0,U} \circ \tau = \rho_{W_0,U_i}$ for some open set W_0 with $W \subset\subset W_0 \subset\subset U_i$. By Lemma 7.4.2(ii) we have that

$$\rho_{W,U_i}(\rho_{U_i,U}^{\text{RO}}(\Phi_n)(\varphi) - \varphi) = \rho_{W,U}(\Phi_n(\tau(\varphi)) - \tau(\varphi))$$

for n large enough. The result now follows from the fact that $\rho_{W,U} \in L(F(U), F(W))$.

(TO)₃ It suffices to show that, for all $f \in E(U_i)$ and $W \subset\subset U_i$, it holds that

$$\rho_{W,U_i}(\rho_{U_i,U}^{\text{RO}}(\Phi_n)(f) - f) \rightarrow 0 \quad \text{in } E(W).$$

Let $\tau \in L(E(U_i), E(U))$ be such that $\rho_{W_0,U} \circ \tau = \rho_{W_0,U_i}$ for some open set W_0 with $W \subset\subset W_0 \subset\subset U_i$. By Lemma 7.4.2(ii) it holds that

$$\rho_{W,U_i}(\rho_{U_i,U}^{\text{RO}}(\Phi_n)(f) - f) = \rho_{W,U}(\Phi_n(\tau(f)) - \tau(f))$$

for n large enough. The result now follows from the fact that $\rho_{W,U} \in L(E(U), E(W))$.

Conversely, assume that $(\rho_{U_i,U}^{\text{RO}}(\Phi_n))_n$ satisfies TO _{j} , $j = 1, 2, 3$, for each i . We shall prove that $(\Phi_n)_n$ does so as well.

(TO)₁ It suffices to show that, for all $f \in E(U)$ and $q \in \text{csn}(F(W))$, with $W \subset\subset U_i$ arbitrary (for some i), there is $\lambda \in \mathcal{A}$ such that

$$\sup_{n \in \mathbb{N}} q(\rho_{W,U}(\Phi_n(f))) / \lambda_n < \infty.$$

By Lemma 7.4.2(iii) we have that

$$\rho_{W,U}(\Phi_n)(f) = \rho_{W,U_i}(\rho_{U_i,U}^{\text{RO}}(\Phi_n)(\rho_{U_i,U}(f)))$$

for n large enough. The result now follows from the fact that $\rho_{W,U_i} \in L(F(U_i), F(W))$.

(TO)₂ It suffices to show that, for all $\varphi \in F(U)$, $q \in \text{csn}(F(W))$, with $W \subset\subset U_i$ arbitrary (for some i), and $\lambda \in \mathcal{I}$ it holds that

$$\sup_{n \in \mathbb{N}} q(\rho_{W,U}(\Phi_n(\varphi) - \varphi)) / \lambda_n < \infty.$$

By Lemma 7.4.2(iii) we have that

$$\rho_{W,U}(\Phi_n(\varphi) - \varphi) = \rho_{W,U_i}(\rho_{U_i,U}^{\text{RO}}(\Phi_n)(\rho_{U_i,U}(\varphi)) - \rho_{U_i,U}(\varphi))$$

for n large enough. The result now follows from the fact that $\rho_{W,U_i} \in L(F(U_i), F(W))$.

(TO)₃ It suffices to show that, for all $f \in E(U)$ and $W \subset\subset U_i$ arbitrary (for some i), it holds that

$$\rho_{W,U}(\Phi_n(f) - f) \rightarrow 0 \quad \text{in } E(W).$$

By Lemma 7.4.2(iii) we have that

$$\rho_{W,U}((\Phi_n)(f) - f) = \rho_{W,U_i}(\rho_{U_i,U}^{\text{RO}}(\Phi_n)(\rho_{U_i,U}(f)) - \rho_{U_i,U}(f))$$

for n large enough. The result now follows from the fact that $\rho_{W,U_i} \in L(E(U_i), E(W))$. \square

The following result is an immediate consequence of Lemmas 7.3.1 and 7.4.5.

Lemma 7.4.7. *Let $U \subseteq X$ be open.*

(i) *Let $\mu^i \in \text{Hom}(E, E)$, $i = 0, \dots, k$, $k \in \mathbb{N}$, be such that $\mu|_F^i \in \text{Hom}(F, F)$ and $\sum_{i=0}^k \mu^i = \text{id}$. Then, $\left(\sum_{i=0}^k \mu_U^i \circ \Phi_{i,n}\right)_n \in \text{TO}_{\text{loc}}(U)$ for all $(\Phi_{i,n})_n \in \text{TO}_{\text{loc}}(U)$, $i = 0, \dots, k$.*

(ii) *Let $\mu \in \text{Hom}(E, E)$ be such that $\mu|_F \in \text{Hom}(F, F)$. Then, $(\mu_U \circ \Phi_n)_n \in \text{TO}_{\text{loc}}^0(U)$ for all $(\Phi_n)_n \in \text{TO}_{\text{loc}}^0(U)$.*

(iii) Let $\mu \in \text{Hom}(E, E)$ be such that $\mu|_F \in \text{Hom}(F, F)$. Then, $(\mu_U \circ \Phi_n - \Phi_n \circ \mu_U)_n \in \text{TO}_{\text{loc}}^0(U)$ for all $(\Phi_n)_n \in \text{TO}_{\text{loc}}(U) \cup \text{TO}_{\text{loc}}^0(U)$.

We conclude this subsection with an important lemma.

Lemma 7.4.8. *Let W, V, U be open sets in X such that $W \subset \subset V \subseteq U$. For every $(\Phi_n)_n \in \text{RO}_{\text{loc}}(V)$ there is $(\Phi'_n)_n \in \text{RO}_{\text{loc}}(U)$ such that*

$$\rho_{W,V} \circ \Phi_n \circ \rho_{V,U} = \rho_{W,U} \circ \Phi'_n$$

for n large enough. Moreover, similar statements holds for TO_{loc} and TO_{loc}^0 .

Proof. We only show the statement for $(\Phi_n)_n \in \text{TO}_{\text{loc}}(V)$; the other cases are similar. Choose open sets W_0, W_1 such that $W \subset \subset W_0 \subset \subset W_1 \subset \subset V$ and let $\mu \in \text{Hom}(E, E)$ be such that $\mu|_F \in \text{Hom}(F, F)$, $\mu_{W_0} = \text{id}$, and $\mu_{U \setminus \overline{W_1}} = 0$. Furthermore, pick an arbitrary element $(\Phi''_n)_n \in \text{TO}_{\text{loc}}(U)$. By Lemma 7.4.6 we have that $\rho_{U \setminus \overline{W_1}, U}^{\text{RO}}([\Phi''_n]) \in \widetilde{\text{TO}}_{\text{loc}}(U \setminus \overline{W_1})$ and by Lemma 7.4.7 it holds that $\mu_V([\Phi_n]) + (\text{id} - \mu)_V(\rho_{V,U}^{\text{RO}}([\Phi''_n])) \in \widetilde{\text{TO}}_{\text{loc}}(V)$. Since

$$\begin{aligned} & \rho_{U \setminus \overline{W_1} \cap V, U \setminus \overline{W_1}}^{\text{RO}}(\rho_{U \setminus \overline{W_1}, U}^{\text{RO}}([\Phi''_n])) \\ &= \rho_{U \setminus \overline{W_1} \cap V, V}^{\text{RO}}(\mu_V([\Phi_n]) + (\text{id} - \mu)_V(\rho_{V,U}^{\text{RO}}([\Phi''_n]))), \end{aligned}$$

Proposition 7.4.4 and Lemma 7.4.6 imply that there is an element $(\Phi'_n)_n \in \text{TO}_{\text{loc}}(U)$ such that

$$\begin{aligned} \rho_{W_0, U}^{\text{RO}}([\Phi'_n]) &= \rho_{W_0, V}^{\text{RO}}(\mu_V([\Phi_n]) + (\text{id} - \mu)_V(\rho_{V,U}^{\text{RO}}([\Phi''_n]))) \\ &= \rho_{W_0, V}^{\text{RO}}([\Phi_n]), \end{aligned}$$

whence the result follows from Lemma 7.4.2(iii). \square

7.4.3 Sheaves of non-linear extensions

Let (E, F) be a test pair of sheaves. For $U \subset X$ open we write $\mathcal{E}(U) = \mathcal{E}(E(U), F(U))$. An element $R \in \mathcal{E}(U)$ is called *local* if

$$\rho_{V,U} \circ \Phi_1 = \rho_{V,U} \circ \Phi_2 \implies \rho_{V,U}(R(\Phi_1)) = \rho_{V,U}(R(\Phi_2))$$

holds for all $V \subseteq U$ open and all $\Phi_1, \Phi_2 \in \text{RO}(U)$. The subset of local elements of $\mathcal{E}(U)$ is denoted by $\mathcal{E}_{\text{loc}}(U)$.

Remark 7.4.9. If $R \in \mathcal{E}(U)$ is local, then the identities $\rho_{V,U} \circ \Phi_1 = \rho_{V,U} \circ \Phi_2$ and $\rho_{V,U} \circ \Psi_{i,1} = \rho_{V,U} \circ \Psi_{i,2}$ with $\Phi_1, \Phi_2, \Psi_{1,i}, \Psi_{2,i} \in \text{RO}(U)$ for $i = 1, \dots, k$ imply that

$$\rho_{V,U}(\text{d}^k R(\Phi_1)(\Psi_{1,1}, \dots, \Psi_{1,k})) = \rho_{V,U}((\text{d}^k R)(\Phi_2)(\Psi_{2,1}, \dots, \Psi_{2,k})).$$

Next, we define a restriction mapping on \mathcal{E}_{loc} .

Lemma 7.4.10. *Let U, V be open subsets of X such that $V \subseteq U$. Then, there is a unique linear mapping $\rho_{V,U}^{\mathcal{E}} : \mathcal{E}_{\text{loc}}(U) \rightarrow \mathcal{E}_{\text{loc}}(V)$ such that:*

- (i) *For all $W \subset\subset V$ and $\Phi \in \text{RO}(V)$, $\Phi' \in \text{RO}(U)$ with $\rho_{W,V} \circ \Phi \circ \rho_{V,U} = \rho_{W,U} \circ \Phi'$ it holds that*

$$\rho_{W,V}(\rho_{V,U}^{\mathcal{E}}(R)(\Phi)) = \rho_{W,U}(R(\Phi')).$$

Moreover, the following properties are satisfied:

- (ii) *For all $W \subset\subset V$ it holds that if $\Phi \in \text{RO}(V)$, $\Phi' \in \text{RO}(U)$ and $\Psi_i \in \text{RO}(V)$, $\Psi'_i \in \text{RO}(U)$, $i = 1, \dots, k$, satisfy*

$$\rho_{W,V} \circ \Phi \circ \rho_{V,U} = \rho_{W,U} \circ \Phi'$$

and

$$\rho_{W,V} \circ \Psi_i \circ \rho_{V,U} = \rho_{W,U} \circ \Psi'_i$$

for $i = 1, \dots, k$, then

$$\begin{aligned} \rho_{W,V}((\text{d}^k(\rho_{V,U}^{\mathcal{E}}(R)))(\Phi)(\Psi_1, \dots, \Psi_k)) \\ = \rho_{W,U}((\text{d}^k R)(\Phi')(\Psi'_1, \dots, \Psi'_k)). \end{aligned}$$

- (iii) *For $W \subseteq V \subseteq U$ it holds that $\rho_{W,V}^{\mathcal{E}} \circ \rho_{V,U}^{\mathcal{E}} = \rho_{W,U}^{\mathcal{E}}$.*

Proof. Let $(V_i)_i$ be an open covering of V such that $V_i \subset\subset V$ for all i and let $(\eta^i)_i \subset \text{Hom}(F|_V, F|_V)$ be a partition of the unity subordinate to $(V_i)_i$. Choose $\tau_i \in L(F(V), F(U))$ such that $\rho_{V_i, U} \circ \tau_i = \rho_{V_i, V}$. For each i we define the mapping $f_i \in L(\text{RO}(V), \text{RO}(U))$ via $f_i(\Phi) = \tau_i \circ \Phi \circ \rho_{V, U}$. Observe that

$$\rho_{V_i, U} \circ f_i(\Phi) = \rho_{V_i, V} \circ \Phi \circ \rho_{V, U}. \quad (7.4.7)$$

Set

$$\rho_{V, U}^{\mathcal{E}}(R) = \sum_i \eta_V^i \circ \rho_{V, U} \circ R \circ f_i.$$

Firstly, we show that $\rho_{V, U}^{\mathcal{E}}(R)$ is smooth. By [101, Lemma 3.8] it suffices to show that $\rho_{W, V} \circ \rho_{V, U}^{\mathcal{E}}(R) : \text{RO}(V) \rightarrow F(W)$ is smooth for all $W \subset\subset V$. Since

$$\rho_{W, V} \circ \rho_{V, U}^{\mathcal{E}}(R) = \sum_{i \in I} \eta_W^i \circ \rho_{W, U} \circ R \circ f_i \quad (7.4.8)$$

for some finite index set I , this follows from the fact that η_W^i , $\rho_{W, U}$, and f_i are continuous linear mappings. Next, we show that $\rho_{V, U}^{\mathcal{E}}(R)$ is local. It suffices to show that for all $W \subset\subset V$ and all $\Phi_1, \Phi_2 \in \text{RO}(V)$ with $\rho_{W, V} \circ \Phi_1 = \rho_{W, V} \circ \Phi_2$ it holds that $\rho_{W, V}(\rho_{V, U}^{\mathcal{E}}(R)(\Phi_1)) = \rho_{W, V}(\rho_{V, U}^{\mathcal{E}}(R)(\Phi_2))$. Suppose that the mapping $\rho_{W, V} \circ \rho_{V, U}^{\mathcal{E}}(R)$ can be represented as (7.4.8). Since $\text{supp } \eta^i \subset V_i$, it suffices to show that $\rho_{W \cap V_i, U}(R(f_i(\Phi_1))) = \rho_{W \cap V_i, U}(R(f_i(\Phi_2)))$ for all $i \in I$. By locality of R this follows from (7.4.7). The linearity of the mapping $\rho_{V, U}^{\mathcal{E}}$ is clear.

(i) Suppose that the mapping $\rho_{W, V} \circ \rho_{V, U}^{\mathcal{E}}(R)$ can be represented as (7.4.8). Since

$$\rho_{W, U}(R(\Phi')) = \sum_{i \in I} \eta_W^i(\rho_{W, U}(R(\Phi'))),$$

we only need to prove that $\rho_{W \cap V_i, U}(R(f_i(\Phi))) = \rho_{W \cap V_i, U}(R(\Phi'))$ for all $i \in I$. As before, this follows from (7.4.7) and the locality of R . The mapping $\rho_{V, U}^{\mathcal{E}}$ is unique because, for all $W \subset\subset V$ and

$\Phi \in \text{RO}(V)$, one can find $\Phi' \in \text{RO}(U)$ such that $\rho_{W,V} \circ \Phi \circ \rho_{V,U} = \rho_{W,U} \circ \Phi'$.

(ii) We use induction on k . The case $k = 0$ has been treated in (i). Now suppose that the statement holds for $k - 1$ and let us show it for k .

$$\begin{aligned}
& \rho_{W,V}((d^k(\rho_{V,U}^\mathcal{E}(R)))(\Phi)(\Psi_1, \dots, \Psi_k)) \\
&= \rho_{W,V} \left(\left. \frac{\partial}{\partial t} \right|_{t=0} (d^{k-1}(\rho_{V,U}^\mathcal{E}(R)))(\Phi + t\Psi_1)(\Psi_2, \dots, \Psi_k) \right) \\
&= \left. \frac{\partial}{\partial t} \right|_{t=0} \rho_{W,V}((d^{k-1}(\rho_{V,U}^\mathcal{E}(R)))(\Phi + t\Psi_1)(\Psi_2, \dots, \Psi_k)) \\
&= \left. \frac{\partial}{\partial t} \right|_{t=0} \rho_{W,V}((d^{k-1}(\rho_{V,U}^\mathcal{E}(R)))(\Phi' + t\Psi'_1)(\Psi'_2, \dots, \Psi'_k)) \\
&= \rho_{W,V} \left(\left. \frac{\partial}{\partial t} \right|_{t=0} (d^{k-1}(\rho_{V,U}^\mathcal{E}(R)))(\Phi' + t\Psi'_1)(\Psi'_2, \dots, \Psi'_k) \right) \\
&= \rho_{W,V}((d^k(\rho_{V,U}^\mathcal{E}(R)))(\Phi')(\Psi'_1, \dots, \Psi'_k)).
\end{aligned}$$

(iii) Let $R \in \mathcal{E}_{\text{loc}}(U)$ be arbitrary. It suffices to show that

$$\rho_{W_0,W}(\rho_{W,V}^\mathcal{E}(\rho_{V,U}^\mathcal{E}(R))(\Phi)) = \rho_{W_0,W}(\rho_{W,U}^\mathcal{E}(R)(\Phi))$$

for all $\Phi \in \text{RO}(W)$ and $W_0 \subset\subset W$. Choose $\Phi' \in \text{RO}(V)$ such that $\rho_{W_0,W} \circ \Phi \circ \rho_{W,V} = \rho_{W_0,V} \circ \Phi'$ and $\Phi'' \in \text{RO}(U)$ such that $\rho_{W_0,V} \circ \Phi' \circ \rho_{V,U} = \rho_{W_0,U} \circ \Phi''$. Hence, also $\rho_{W_0,W} \circ \Phi \circ \rho_{W,U} = \rho_{W_0,U} \circ \Phi''$. Therefore, (i) implies that

$$\begin{aligned}
& \rho_{W_0,W}(\rho_{W,V}^\mathcal{E}(\rho_{V,U}^\mathcal{E}(R))(\Phi)) = \rho_{W_0,V}(\rho_{V,U}^\mathcal{E}(R)(\Phi')) \\
&= \rho_{W_0,U}(R(\Phi'')) = \rho_{W_0,W}(\rho_{W,U}^\mathcal{E}(R)(\Phi)).
\end{aligned}$$

□

We now discuss the extension of sheaf morphisms to \mathcal{E} .

Lemma 7.4.11. *Let $T : F \times \dots \times F \rightarrow F$ be a multilinear sheaf morphism and let $U \subseteq X$ be open. Consider the mapping $\tilde{T}_U :$*

$\mathcal{E}(U) \times \cdots \times \mathcal{E}(U) \rightarrow \mathcal{E}(U)$ given by $\widetilde{T}_U := \widetilde{T}_U$ as in (7.2.1). Then, \widetilde{T}_U preserves locality, i.e. $\widetilde{T}_U(\mathcal{E}_{\text{loc}}(U), \dots, \mathcal{E}_{\text{loc}}(U)) \subseteq \mathcal{E}_{\text{loc}}(U)$ and

$$\rho_{V,U}^{\mathcal{E}}(\widetilde{T}_U(R_1, \dots, R_k)) = \widetilde{T}_V(\rho_{V,U}^{\mathcal{E}}(R_1), \dots, \rho_{V,U}^{\mathcal{E}}(R_k))$$

for all $V \subseteq U$ open.

Proof. The mappings \widetilde{T}_U are well-defined by Lemma 7.2.1. Moreover, the fact that the \widetilde{T}_U preserve locality is clear from their definition. In order to show the last property, it suffices to show that

$$\begin{aligned} & \rho_{W,V}(\rho_{V,U}^{\mathcal{E}}(\widetilde{T}_U(R_1, \dots, R_k))(\Phi)) \\ &= \rho_{W,V}(\widetilde{T}_V(\rho_{V,U}^{\mathcal{E}}(R_1), \dots, \rho_{V,U}^{\mathcal{E}}(R_k))(\Phi)) \end{aligned}$$

for all $\Phi \in \text{RO}(V)$ and $W \subset \subset V$. Choose $\Phi' \in \text{RO}(U)$ such that $\rho_{W,V} \circ \Phi \circ \rho_{V,U} = \rho_{W,U} \circ \Phi'$. Lemma 7.4.10(i) implies that

$$\begin{aligned} & \rho_{W,V}(\rho_{V,U}^{\mathcal{E}}(\widetilde{T}_U(R_1, \dots, R_k))(\Phi)) \\ &= \rho_{W,U}(\widetilde{T}_U(R_1, \dots, R_k)(\Phi')) \\ &= \rho_{W,U}(T_U(R_1(\Phi'), \dots, R_k(\Phi'))) \\ &= T_W(\rho_{W,U}(R_1(\Phi')), \dots, \rho_{W,U}(R_k(\Phi'))) \\ &= T_W(\rho_{W,V}(\rho_{V,U}^{\mathcal{E}}(R_1)(\Phi)), \dots, \rho_{W,V}(\rho_{V,U}^{\mathcal{E}}(R_k)(\Phi))) \\ &= \rho_{W,V}(T_V(\rho_{V,U}^{\mathcal{E}}(R_1)(\Phi), \dots, \rho_{V,U}^{\mathcal{E}}(R_k)(\Phi))) \\ &= \rho_{W,V}(\widetilde{T}_V(\rho_{V,U}^{\mathcal{E}}(R_1), \dots, \rho_{V,U}^{\mathcal{E}}(R_k))(\Phi)). \end{aligned}$$

□

Lemma 7.4.12. *Let $T : E \rightarrow E$ be a sheaf morphism such that its restriction $T|_F : F \rightarrow F$ is also a sheaf morphism and let $U \subseteq X$ be open. Consider the mapping $\widehat{T}_U : \mathcal{E}(U) \rightarrow \mathcal{E}(U)$ given by $\widehat{T}_U := \widehat{T}_U$ as in (7.2.4). Then, \widehat{T}_U preserves locality and*

$$\rho_{V,U}^{\mathcal{E}}(\widehat{T}_U(R)) = \widehat{T}_V(\rho_{V,U}^{\mathcal{E}}(R))$$

for all $V \subseteq U$ open.

Proof. The mappings \widehat{T}_U are well-defined by Lemma 7.2.4. Next, we show that \widehat{T}_U preserves locality. Let $V \subseteq U$ be open, $R \in \mathcal{E}_{\text{loc}}(U)$, and $\Phi_1, \Phi_2 \in \text{RO}(U)$ be such that $\rho_{V,U} \circ \Phi_1 = \rho_{V,U} \circ \Phi_2$. We have that

$$\begin{aligned}
& \rho_{V,U}((\widehat{T}_U R)(\Phi_1)) \\
&= \rho_{V,U}(T_U(R(\Phi_1)) - \text{d}R(\Phi_1)(T_U \circ \Phi_1 - \Phi_1 \circ T_U)) \\
&= T_V(\rho_{V,U}(R(\Phi_1))) - \rho_{V,U}(\text{d}R(\Phi_1)(T_U \circ \Phi_1 - \Phi_1 \circ T_U)) \\
&= T_V(\rho_{V,U}(R(\Phi_2))) - \rho_{V,U}(\text{d}R(\Phi_2)(T_U \circ \Phi_2 - \Phi_2 \circ T_U)) \\
&= \rho_{V,U}((\widehat{T}_U R)(\Phi_2))
\end{aligned}$$

because

$$\begin{aligned}
\rho_{V,U} \circ (T_U \circ \Phi_1 - \Phi_1 \circ T_U) &= T_V \circ \rho_{V,U} \circ \Phi_1 - \rho_{V,U} \circ \Phi_1 \circ T_U \\
&= T_V \circ \rho_{V,U} \circ \Phi_2 - \rho_{V,U} \circ \Phi_2 \circ T_U = \rho_{V,U} \circ (T_U \circ \Phi_2 - \Phi_2 \circ T_U).
\end{aligned}$$

For the second statement, let $W \subset\subset V$ and $\Phi \in \text{RO}(V)$ be arbitrary. Choose $\Phi' \in \text{RO}(U)$ such that $\rho_{W,V} \circ \Phi \circ \rho_{V,U} = \rho_{W,U} \circ \Phi'$. Then,

$$\begin{aligned}
& \rho_{W,V}(\rho_{V,U}^{\mathcal{E}}(\widehat{T}_U(R))(\Phi)) \\
&= \rho_{W,U}(\widehat{T}_U(R)(\Phi')) \\
&= \rho_{W,U}(T_U(R(\Phi')) - \text{d}R(\Phi')(T_U \circ \Phi' - \Phi' \circ T_U)) \\
&= T_W(\rho_{W,U}(R(\Phi'))) - \rho_{W,V}(\text{d}(\rho_{V,U}^{\mathcal{E}} R)(\Phi)(T_V \circ \Phi - \Phi \circ T_V)) \\
&= \rho_{W,V}(T_V((\rho_{V,U}^{\mathcal{E}} R)(\Phi)) - \text{d}(\rho_{V,U}^{\mathcal{E}} R)(\Phi)(T_V \circ \Phi - \Phi \circ T_V)) \\
&= \rho_{W,V}(\widehat{T}_V(\rho_{V,U}^{\mathcal{E}} R)(\Phi)).
\end{aligned}$$

□

We now turn to the quotient construction. Let sc be an admissible pair of scales. For $U \subseteq X$ open we define the space of moderate elements of $\mathcal{E}_{\text{loc}}(U)$ (with respect to sc) as

$$\begin{aligned}
& \mathcal{E}_{\mathcal{M},\text{loc}}(U) = \mathcal{E}_{\mathcal{M},\text{loc}}(U, \text{sc}) := \\
& \mathcal{E}_{\mathcal{M}}(E(U), F(U), \text{TO}_{\text{loc}}(U), \text{TO}_{\text{loc}}^0(U), \text{sc}) \cap \mathcal{E}_{\text{loc}}(U),
\end{aligned}$$

and the space of negligible elements (with respect to sc) as

$$\begin{aligned}\mathcal{E}_{\mathcal{N},\text{loc}}(U) &= \mathcal{E}_{\mathcal{N},\text{loc}}(U, \text{sc}) := \\ \mathcal{E}_{\mathcal{N}}(E(U), F(U), \text{TO}_{\text{loc}}(U), \text{TO}_{\text{loc}}^0(U), \text{sc}) \cap \mathcal{E}_{\text{loc}}(U).\end{aligned}$$

We set

$$\mathcal{G}_{\text{loc}}(U) = \mathcal{G}_{\text{loc}}(U, \text{sc}) := \mathcal{E}_{\mathcal{M},\text{loc}}(U) / \mathcal{E}_{\mathcal{N},\text{loc}}(U).$$

Lemma 7.4.13. *Let $U \subseteq X$ be open and let $(U_i)_i$ be an open covering of U . Let $R \in \mathcal{E}_{\text{loc}}(U)$. Then, R is moderate (negligible, respectively) if and only if $\rho_{U_i,U}^{\mathcal{E}}(R)$ is moderate (negligible, respectively) for all i .*

Proof. Let $R \in \mathcal{E}_{\text{loc}}(U)$ be moderate or negligible. The moderateness or negligibility of $\rho_{U_i,U}^{\mathcal{E}}(R)$ is determined by

$$q(\rho_{W,U_i}((d^k(\rho_{U_i,U}^{\mathcal{E}}(R)))(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n})))$$

for n large enough, where $k \in \mathbb{N}$, $(\Phi_n)_n \in \text{TO}_{\text{loc}}(U_i)$, $(\Psi_{j,n})_n \in \text{TO}_{\text{loc}}^0(U_i)$, $j = 1, \dots, k$, $W \subset \subset U_i$, and $q \in \text{csn}(F(W))$ are arbitrary. By Lemma 7.4.8 there are $(\Phi'_n)_n \in \text{TO}_{\text{loc}}(U)$ and $(\Psi'_{j,n})_n \in \text{TO}_{\text{loc}}^0(U)$ such that $\rho_{W,U_i} \circ \Phi_n \circ \rho_{U_i,U} = \rho_{W,U} \circ \Phi'_n$ and $\rho_{W,U_i} \circ \Psi_{j,n} \circ \rho_{U_i,U} = \rho_{W,U} \circ \Psi'_{j,n}$ for $j = 1, \dots, k$ and n large enough. Hence, Lemma 7.4.10(ii) implies that

$$\begin{aligned}\rho_{W,U_i}((d^k(\rho_{U_i,U}^{\mathcal{E}}(R)))(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n})) \\ = \rho_{W,U}((d^k R)(\Phi'_n)(\Psi'_{1,n}, \dots, \Psi'_{k,n}))\end{aligned}$$

for n large enough. The moderateness or negligibility of $\rho_{U_i,U}^{\mathcal{E}}(R)$ therefore follows from the corresponding property of R and the continuity of $\rho_{W,U}$. Conversely, suppose that $\rho_{U_i,U}^{\mathcal{E}}(R)$ is moderate or negligible for all i . The moderateness of R is determined by

$$q(\rho_{W,U}((d^k R)(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n})))$$

for n large enough, where $k \in \mathbb{N}$, $(\Phi_n)_n \in \text{TO}_{\text{loc}}(U)$, $(\Psi_{j,n})_n \in \text{TO}_{\text{loc}}^0(U)$, $j = 1, \dots, k$, $W \subset\subset U_i$ (for some i), and $q \in \text{csn}(F(W))$ are arbitrary. Lemma 7.4.2(iii) and Lemma 7.4.10(ii) imply that

$$\begin{aligned} & \rho_{W,U}((d^k R)(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{l,n})) \\ &= \rho_{W,U_i}((d^k(\rho_{U_i,U}^\mathcal{E}(R)))(\rho_{U_i,U}^{\text{RO}}(\Phi_n))(\rho_{U_i,U}^{\text{RO}}(\Psi_{1,n}), \dots, \rho_{U_i,U}^{\text{RO}}(\Psi_{k,n}))) \end{aligned}$$

for n large enough. The moderateness or negligibility of R therefore follows from the corresponding property of $\rho_{U_i,U}^\mathcal{E}(R)$ and the continuity of ρ_{W,U_i} . \square

Lemmas 7.4.10 and 7.4.13 imply that the mappings

$$\rho_{V,U}^\mathcal{G}([R]) := [\rho_{V,U}^\mathcal{E}(R)]$$

define a presheaf structure on \mathcal{G}_{loc} . We now show that it is in fact a sheaf.

Proposition 7.4.14. *\mathcal{G}_{loc} is a sheaf.*

Proof. (S1) Immediate consequence of Lemma 7.4.13.

(S2) Let $U \subseteq X$ be open and let $(U_i)_i$ be an open covering of U . We may assume that $U_i \subset\subset U$ for all i . Suppose that $[R_i] \in \mathcal{G}_{\text{loc}}(U_i)$ are given such that $\rho_{U_i \cap U_j, U_i}^\mathcal{G}([R_i]) = \rho_{U_i \cap U_j, U_j}^\mathcal{G}([R_j])$ for all i, j . Let $(\eta^i)_i \subset \text{Hom}(F|_U, F|_U)$ be a partition of the unity subordinate to $(U_i)_i$. Choose $\tau_i \in L(F(U_i), F(U))$ such that $\rho_{V_i,U} \circ \tau_i = \rho_{V_i,U_i}$ for some $V_i \subset\subset U_i$ with $\text{supp } \eta^i \subset V_i$. We define

$$R = \sum_i \eta_U^i \circ \tau_i \circ R_i \circ \rho_{U_i,U}^{\text{RO}}.$$

We start by showing that $R \in C^\infty(\text{RO}(U), F(U))$. By [101, Lemma 3.8] it suffices to show that $\rho_{W,U} \circ R : \text{RO}(U) \rightarrow F(W)$ is smooth for all $W \subset\subset U$. Since

$$\rho_{W,U} \circ R = \sum_{i \in I} \eta_W^i \circ \rho_{W,U} \circ \tau_i \circ R_i \circ \rho_{U_i,U}^{\text{RO}} \quad (7.4.9)$$

for some finite index set I , the smoothness of this mapping follows from the fact that the linear mappings η_W^i , $\rho_{W,U}$, τ_i , and $\rho_{U_i,U}^{\text{RO}}$ are continuous (Lemma 7.4.2). Next, we show that R is local. We need to prove that

$$\rho_{W,U}(R(\Phi_1)) = \rho_{W,U}(R(\Phi_2))$$

for all $W \subset\subset U$ and $\Phi_1, \Phi_2 \in \text{RO}(U)$ with $\rho_{W,U} \circ \Phi_1 = \rho_{W,U} \circ \Phi_2$. Suppose that the mapping $\rho_{W,U} \circ R$ can be represented as (7.4.9). Since $\text{supp } \eta^i \subset V_i$, it suffices to show that

$$\rho_{W \cap V_i, U_i}(R_i(\rho_{U_i,U}^{\text{RO}}(\Phi_1))) = \rho_{W \cap V_i, U_i}(R_i(\rho_{U_i,U}^{\text{RO}}(\Phi_2)))$$

for all $i \in I$. By locality of R_i this follows from Lemma 7.4.2(iv). We proceed with showing that R is moderate. The moderateness of R is determined by

$$q(\rho_{W,U}((d^k R)(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n})))$$

for n large enough, where $k \in \mathbb{N}$, $(\Phi_n)_n \in \text{TO}_{\text{loc}}(U)$, $(\Psi_{j,n})_n \in \text{TO}_{\text{loc}}^0(U)$, $j = 1, \dots, k$, $W \subset\subset U$, and $q \in \text{csn}(F(W))$ are arbitrary. Since

$$\begin{aligned} & \rho_{W,U}((d^k R)(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n})) \\ &= (d^k(\rho_{W,U} \circ R))(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n}) \\ &= \left(d^k \left(\sum_{i \in I} \eta_W^i \circ \rho_{W,U} \circ \tau_i \circ R_i \circ \rho_{U_i,U}^{\text{RO}} \right) \right) (\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n}) \\ &= \sum_{i \in I} (\eta_W^i \circ \rho_{W,U} \circ \tau_i)((d^k R_i)(\rho_{U_i,U}^{\text{RO}}(\Phi_n))) \\ & \quad (\rho_{U_i,U}^{\text{RO}}(\Psi_{1,n}), \dots, \rho_{U_i,U}^{\text{RO}}(\Psi_{k,n})) \end{aligned}$$

for some finite index set I , the moderateness of R follows from the continuity of the mapping $\eta_W^i \circ \rho_{W,U} \circ \tau_i$ and the moderateness of the R_i . Finally, we show that $\rho_{U_i,U}^{\mathcal{G}}([R]) = [R_i]$ for all i . We need to show that $\rho_{U_i,U}^{\mathcal{E}}(R) - R_i$ is negligible. The negligibility is determined by

$$q(\rho_{W,U_i}((d^k(\rho_{U_i,U}^{\mathcal{E}}(R) - R_i))(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n})))$$

for n large enough, where $k \in \mathbb{N}$, $(\Phi_n)_n \in \text{TO}_{\text{loc}}(U_i)$, $(\Psi_{j,n})_n \in \text{TO}_{\text{loc}}^0(U_i)$, $j = 1, \dots, k$, $W \subset\subset U_i$, and $q \in \text{csn}(F(W))$ are arbitrary. By Lemma 7.4.8 there are $(\Phi'_n)_n \in \text{TO}_{\text{loc}}(U)$, $(\Psi'_{j,n})_n \in \text{TO}_{\text{loc}}^0(U)$ for $j = 1, \dots, k$ such that $\rho_{W,U_i} \circ \Phi_n \circ \rho_{U_i,U} = \rho_{W,U} \circ \Phi'_n$ and $\rho_{W,U_i} \circ \Psi_{j,n} \circ \rho_{U_i,U} = \rho_{W,U} \circ \Psi'_{j,n}$ for $j = 1, \dots, k$ and n large enough. Hence, Lemma 7.4.2(ii) yields that

$$\begin{aligned}
& \rho_{W,U_i}((d^k(\rho_{U_i,U}^{\mathcal{E}}(R)))(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n})) \\
&= \rho_{W,U}((d^k R)(\Phi'_n)(\Psi'_{1,n}, \dots, \Psi'_{k,n})) \\
&= (d^k(\rho_{W,U} \circ R))(\Phi'_n)(\Psi'_{1,n}, \dots, \Psi'_{k,n}) \\
&= \left(d^k \left(\sum_{j \in I} \eta_W^j \circ \rho_{W,U} \circ \tau_j \circ R_j \circ \rho_{U_j,U}^{\text{RO}} \right) \right) (\Phi'_n)(\Psi'_{1,n}, \dots, \Psi'_{k,n}) \\
&= \sum_{j \in I} \eta_W^j (\rho_{W,U}(\tau_j((d^k R_j)(\rho_{U_j,U}^{\text{RO}}(\Phi'_n)) \cdot \\
&\quad (\rho_{U_j,U}^{\text{RO}}(\Psi'_{1,n}), \dots, \rho_{U_j,U}^{\text{RO}}(\Psi'_{k,n}))))))
\end{aligned}$$

for n large enough. On the other hand, Lemma 7.4.2(ii) and the fact that $(\Phi_n)_n$ is localizing imply that $\rho_{W,U_i} \circ \Phi_n = \rho_{W,U_i} \circ \rho_{U_i,U}^{\text{RO}}(\Phi'_n)$ and $\rho_{W,U_i} \circ \Psi_{j,n} = \rho_{W,U_i} \circ \rho_{U_i,U}^{\text{RO}}(\Psi'_{j,n})$ for $j = 1, \dots, k$ and n large enough. By Remark 7.4.9 we obtain that

$$\begin{aligned}
& \rho_{W,U_i}((d^k R_i)(\Phi_n)(\Psi_{1,n}, \dots, \Psi_{k,n})) \\
&= \rho_{W,U_i}((d^k R_i)(\rho_{U_i,U}^{\text{RO}}(\Phi'_n))(\rho_{U_i,U}^{\text{RO}}(\Psi'_{1,n}), \dots, \rho_{U_i,U}^{\text{RO}}(\Psi'_{k,n}))) \\
&= \sum_{j \in I} \eta_W^j (\rho_{W,U_i}((d^k R_i)(\rho_{U_i,U}^{\text{RO}}(\Phi'_n))(\rho_{U_i,U}^{\text{RO}}(\Psi'_{1,n}), \dots, \rho_{U_i,U}^{\text{RO}}(\Psi'_{k,n}))))
\end{aligned}$$

for n large enough. Since $\text{supp } \eta^j \subset V_j$ and $\rho_{V_j,U} \circ \tau_j = \rho_{V_j,U_j}$, it suffices to estimate

$$\begin{aligned}
& \rho_{W \cap V_j, U_j}((d^k R_j)(\rho_{U_j,U}^{\text{RO}}(\Phi'_n))(\rho_{U_j,U}^{\text{RO}}(\Psi'_{1,n}), \dots, \rho_{U_j,U}^{\text{RO}}(\Psi'_{k,n}))) \\
& - \rho_{W \cap V_j, U_i}((d^k R_i)(\rho_{U_i,U}^{\text{RO}}(\Phi'_n))(\rho_{U_i,U}^{\text{RO}}(\Psi'_{1,n}), \dots, \rho_{U_i,U}^{\text{RO}}(\Psi'_{k,n})))
\end{aligned}$$

for all $j \in I$. By Lemma 7.4.10(ii) we have that

$$\begin{aligned} & \rho_{W \cap V_j, U_j}((d^k R_j)(\rho_{U_j, U}^{\text{RO}}(\Phi'_n))(\rho_{U_j, U}^{\text{RO}}(\Psi'_{1,n}), \dots, \rho_{U_j, U}^{\text{RO}}(\Psi'_{k,n}))) \\ &= \rho_{W \cap V_j, U_i \cap U_j}((d^k(\rho_{U_i \cap U_j, U_j}^{\mathcal{E}}(R_j)))(\rho_{U_i \cap U_j, U}^{\text{RO}}(\Phi'_n))). \\ & (\rho_{U_i \cap U_j, U}^{\text{RO}}(\Psi'_{1,n}), \dots, \rho_{U_i \cap U_j, U}^{\text{RO}}(\Psi'_{k,n}))) \end{aligned}$$

and

$$\begin{aligned} & \rho_{W \cap V_j, U_i}((d^k R_i)(\rho_{U_i, U}^{\text{RO}}(\Phi'_n))(\rho_{U_i, U}^{\text{RO}}(\Psi'_{1,n}), \dots, \rho_{U_i, U}^{\text{RO}}(\Psi'_{k,n}))) \\ &= \rho_{W \cap V_j, U_i \cap U_j}((d^k(\rho_{U_i \cap U_j, U_i}^{\mathcal{E}}(R_i)))(\rho_{U_i \cap U_j, U}^{\text{RO}}(\Phi'_n))). \\ & (\rho_{U_i \cap U_j, U}^{\text{RO}}(\Psi'_{1,n}), \dots, \rho_{U_i \cap U_j, U}^{\text{RO}}(\Psi'_{k,n}))). \end{aligned}$$

The negligibility now follows from the assumption. \square

Next, we discuss the embedding of E into \mathcal{G}_{loc} . For $U \subseteq X$ open we consider the canonical embeddings $\iota_U : E(U) \rightarrow \mathcal{E}(U)$ and $\sigma_U : F(U) \rightarrow \mathcal{E}(U)$. Clearly, $\iota_U(E(U)) \subseteq \mathcal{E}_{\text{loc}}(U)$ and $\sigma_U(F(U)) \subseteq \mathcal{E}_{\text{loc}}(U)$. Hence, Proposition 7.3.2 implies that the mappings

$$\iota_U : E(U) \rightarrow \mathcal{G}_{\text{loc}}(U) : f \rightarrow [\iota_U(f)]$$

and

$$\sigma_U : F(U) \rightarrow \mathcal{G}_{\text{loc}}(U) : \varphi \rightarrow [\sigma_U(\varphi)]$$

are linear embeddings such that $\iota_{U|F(U)} = \sigma_U$.

Proposition 7.4.15. *The embeddings $\iota : E \rightarrow \mathcal{G}_{\text{loc}}$ and $\sigma : F \rightarrow \mathcal{G}_{\text{loc}}$ are sheaf monomorphisms such that $\iota|_F = \sigma$.*

Proof. We already noticed that $\iota_{U|F(U)} = \sigma_U$ for all $U \subseteq X$ open. Since F is a subsheaf of E , it therefore suffices to show that ι is a sheaf morphism. Let U, V be open subsets of X such that $V \subseteq U$. We need to show that $\rho_{V, U}^{\mathcal{E}}(\iota_U(f)) - \iota_V(\rho_{V, U}(f))$ is negligible for all $f \in E(U)$. In fact, we shall prove the ensuing stronger assertion: For all $W \subset\subset V$ and $(\Phi_n)_n \in \text{RO}_{\text{loc}}(V)$ it holds that

$$\rho_{W, V}(\rho_{V, U}^{\mathcal{E}}(\iota_U(f))(\Phi_n)) = \rho_{W, V}(\iota_V(\rho_{V, U}(f))(\Phi_n))$$

for n large enough. By Lemma 7.4.8 there is $(\Phi'_n)_n \in \text{RO}_{\text{loc}}(U)$ such that $\rho_{W,V} \circ \Phi_n \circ \rho_{V,U} = \rho_{W,U} \circ \Phi'_n$. Hence, Lemma 7.4.10(i) yields that

$$\begin{aligned} \rho_{W,V}(\rho_{V,U}^{\mathcal{E}}(\iota_U(f))(\Phi_n)) &= \rho_{W,U}(\iota_U(f)(\Phi'_n)) = \rho_{W,U}(\Phi'_n(f)) \\ &= \rho_{W,V}(\Phi_n(\rho_{V,U}(f))) = \rho_{W,V}(\iota_V(\rho_{V,U}(f))(\Phi_n)) \end{aligned}$$

for n large enough. \square

We end this section by showing how one can extend sheaf morphisms to \mathcal{G}_{loc} .

Lemma 7.4.16. *Let $T : F \times \cdots \times F \rightarrow F$ be a multilinear sheaf morphism. The mappings $\tilde{T}_U : \mathcal{G}_{\text{loc}}(U) \times \cdots \times \mathcal{G}_{\text{loc}}(U) \rightarrow \mathcal{G}_{\text{loc}}(U)$ given by*

$$\tilde{T}_U([R_1], \dots, [R_k]) := [\tilde{T}_U(R_1, \dots, R_k)]$$

are well-defined and multilinear such that

$$\tilde{T}_U(\sigma_U(\varphi_1), \dots, \sigma_U(\varphi_k)) = \sigma_U(T_U(\varphi_1, \dots, \varphi_k)).$$

Moreover, \tilde{T} is a multilinear sheaf morphism.

Lemma 7.4.17. *Let $T : E \rightarrow E$ be a sheaf morphism such that its restriction $T|_F : F \rightarrow F$ is also a sheaf morphism. Then, the mappings $\hat{T}_U : \mathcal{G}_{\text{loc}}(U) \rightarrow \mathcal{G}_{\text{loc}}(U)$ given by*

$$\hat{T}_U[R] := [\hat{T}_U R]$$

are well-defined and satisfy $\hat{T}_U \circ \iota_U = \iota_U \circ T_U$. Moreover, \hat{T} is a sheaf morphism.

Corollary 7.4.18. *For all $U \subseteq X$ open the sheaf $\mathcal{G}_{\text{loc}|U}$ is fine.*

Proof. Let A and B be closed sets in U such that $A \cap B = \emptyset$. Let $\tau \in \text{Hom}(F|_U, F|_U)$ be such that $\tau_V = \text{id}$ and $\tau_W = 0$ for some open neighbourhoods V and W (in U) of A and B , respectively. Consider the associated sheaf morphism $\tau \in \text{Hom}(\mathcal{G}_{\text{loc}|U}, \mathcal{G}_{\text{loc}|U})$. Then,

$$\tau_V([R]) = [\tau_V(R)] = [\tau_V \circ R] = [R]$$

for all $[R] \in \mathcal{G}_{\text{loc}}(V)$. Similarly, one can show that $\tau_W = 0$. \square

Corollary 7.4.19. *Suppose that F is a locally convex sheaf of algebras. Then, \mathcal{G}_{loc} is a sheaf of algebras and σ is a sheaf monomorphism of algebras.*

7.5 Diffeomorphism invariant differential algebras containing spaces of ultradistributions

We now apply the general theory developed in Sections 7.2-7.4 to construct Colombeau algebras containing spaces of ultradistributions that are invariant under real analytic diffeomorphisms. It is important to point out that our construction in Chapter 4 was given in the context of special Colombeau algebras and therefore cannot be diffeomorphism invariant. In order to not having to develop the theory of ultradistributions on manifolds here, we restrict our considerations to the local case, i.e. to open subsets of \mathbb{R}^d , where diffeomorphism invariance can be stated easily. Throughout this section we fix a weight sequence M_p satisfying (M.1), (M.2), and (M.3)'.

The existence of \mathcal{D}^* -partitions of the unity implies that the pair $(\mathcal{D}'^*, \mathcal{E}^*)$ is a test pair of sheaves on \mathbb{R}^d . We shall employ the ensuing asymptotic scales:

$$\begin{aligned}\mathcal{A}^{(M_p)} &:= \{e^{M(\lambda n)} \mid \lambda > 0\}, & \mathcal{I}^{(M_p)} &:= \{e^{-M(\lambda n)} \mid \lambda > 0\}, \\ \mathcal{A}^{\{M_p\}} &:= \{e^{M_{\lambda_j}(n)} \mid \lambda_j \in \mathfrak{R}\}, & \mathcal{I}^{\{M_p\}} &:= \{e^{-M_{\lambda_j}(n)} \mid \lambda_j \in \mathfrak{R}\}.\end{aligned}$$

Condition (M.2) (and Lemma 2.2.3 in the Roumieu case) ensure that $\text{sc}^* := (\mathcal{A}^*, \mathcal{I}^*)$ is an admissible pair of scales. For $\Omega \subseteq \mathbb{R}^d$ open we define

$$\text{TO}_{\text{loc}}^*(\Omega) := \text{TO}_{\text{loc}}(\Omega, \mathcal{D}'^*, \mathcal{E}'^*, \text{sc}^*)$$

$$\text{TO}_{\text{loc}}^{0,*}(\Omega) := \text{TO}_{\text{loc}}^0(\Omega, \mathcal{D}'^*, \mathcal{E}'^*, \text{sc}^*).$$

In the next lemma, we denote by $\mathcal{E}_{\mathcal{M}}(\Omega)$ and $\mathcal{E}_{\mathcal{N}}(\Omega)$ the spaces of moderate and negligible sequences of ultradifferentiable functions of class $*$, respectively, introduced in Section 3.4.

Lemma 7.5.1. *Let $(\Phi_n)_n \in L(\mathcal{D}'^*(\Omega), \mathcal{E}^*(\Omega))^{\mathbb{N}}$. Then, $(\Phi_n)_n$ satisfies $(\text{TO})_1$ ($(\text{TO})_2$, respectively) if and only if $(\Phi_n(f))_n \in \mathcal{E}_{\mathcal{M}}(\Omega)$ for all $f \in \mathcal{D}'^*(\Omega)$ ($(\Phi_n(\varphi) - \varphi)_n \in \mathcal{E}_{\mathcal{N}}(\Omega)$ for all $\varphi \in \mathcal{E}^*(\Omega)$, respectively).*

Proof. In the Beurling case, this follows directly from the definitions while, in the Roumieu case, this is a consequence of the projective descriptions of the spaces $\mathcal{E}^{\{M_p\}}(\Omega)$ [92, Prop. 3.5] and $s^{\{M_p\}}$ (Example 3.2.2). \square

For the results from Sections 7.2-7.4 to be applicable in the present situation, we must show that $\text{TO}_{\text{loc}}^*(\Omega)$ is non-empty for every open set $\Omega \subseteq \mathbb{R}^d$. To this end, we shall use the same approach as in [45] and modify our ideas from Section 4.2. We start with the following lemma.

Lemma 7.5.2. *Let M_p and N_p be two weight sequences satisfying (M.1) such that $N_p \prec M_p$. Then, there is a decreasing sequence $(r_n)_n$ of positive numbers with $\lim_{n \rightarrow \infty} r_n = 0$ such that for every $\lambda > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$M(t) \leq N(r_n t) + M(\lambda n), \quad t \geq 0.$$

Proof. By [89, Lemma 3.10] there is an increasing continuous function $\varepsilon : [0, \infty) \rightarrow (0, \infty)$ satisfying $\lim_{t \rightarrow \infty} \varepsilon(t)/t = 0$ such that $M(t) = N(\varepsilon(t))$ for all $t \geq 0$. The sequence

$$r_n := \sup_{t \geq \sqrt{n}} \frac{\varepsilon(t)}{t}, \quad n \in \mathbb{N},$$

satisfies all requirements. \square

By [89, Lemma 4.3] there is a weight sequence N_p satisfying (M.1) and (M.3)' such that $N_p \prec M_p$. Pick $\psi \in \mathcal{D}^{(M_p)}(\mathbb{R}^d)$ even with $0 \leq \psi \leq 1$, $\text{supp } \psi \subseteq \overline{B}(0, 2)$, and $\psi \equiv 1$ on $\overline{B}(0, 1)$, and $\chi \in \mathcal{D}^{(N_p)}(\mathbb{R}^d)$ even with $\text{supp } \chi \subseteq \overline{B}(0, 2)$ and $\chi \equiv 1$ on $\overline{B}(0, 1)$. Choose $(r_n)_n$ as in Lemma 7.5.2. We define $\rho_n = n^d \mathcal{F}^{-1}(\psi)(n \cdot) \chi(\cdot / r_n)$. Next, let $(K_n)_n$ be an exhaustion by compact sets of Ω and choose $\kappa_n \in \mathcal{D}^{(M_p)}(\Omega)$ such that $\kappa_n \equiv 1$ on K_n . Finally, we define

$$\Phi_n(f) = (\kappa_n f) * \rho_n = \langle f(x), \kappa_n(x) \rho_n(\cdot - x) \rangle, \quad f \in \mathcal{D}'^*(\Omega).$$

Observe that $(\Phi_n)_n \in L(\mathcal{D}'^*(\Omega), \mathcal{E}^*(\Omega))^{\mathbb{N}}$ by [89, Prop. 6.10]. Our aim is to show that $(\Phi_n)_n \in \text{TO}_{\text{loc}}^*(\Omega)$. We need the following lemma.

Lemma 7.5.3.

(i) For all $n \in \mathbb{N}$ it holds that

$$\sup_{\xi \in \mathbb{R}^d} |\widehat{\rho}_n(\xi)| \leq \frac{1}{(2\pi)^d} \|\widehat{\chi}\|_{L^1}.$$

(ii) For all $h, \lambda > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\sup_{n \geq n_0} \sup_{|\xi| \geq 4n} |\widehat{\rho}_n(\xi)| e^{M(\xi/h) - M(\lambda n)} < \infty.$$

(iii) For all $\lambda > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\sup_{n \geq n_0} \sup_{|\xi| \leq n/2} |1 - \widehat{\rho}_n(\xi)| e^{M(\lambda n)} < \infty.$$

Proof. (i) Obvious.

(ii) Let $n \in \mathbb{N}$ be arbitrary. We have that

$$\begin{aligned} |\widehat{\rho}_n(\xi)| &= \frac{r_n^d}{(2\pi)^d} \left| \int_{\mathbb{R}^d} \psi(\eta/n) \widehat{\chi}(r_n(\xi - \eta)) d\eta \right| \\ &\leq \frac{r_n^d}{(2\pi)^d} \int_{|\eta| \leq 2n} |\widehat{\chi}(r_n(\xi - \eta))| d\eta \\ &= \frac{1}{(2\pi)^d} \int_{|\xi - \frac{\eta}{r_n}| \leq 2n} |\widehat{\chi}(\eta)| d\eta. \end{aligned}$$

Lemma 2.2.9 implies that $|\widehat{\chi}(\eta)| \leq Ce^{-N(2H\eta/h)}$, $\eta \in \mathbb{R}^d$, for some $C > 0$. Furthermore, notice that $|\eta| \geq r_n|\xi|/2$ for all $\xi, \eta \in \mathbb{R}^d$ with $|\xi| \geq 4n$ and $|\xi - \eta/r_n| \leq 2n$. Hence, we obtain that $|\widehat{\rho}_n(\xi)| \leq C'e^{-N(r_n\xi/h)}$ for all $\xi \in \mathbb{R}^d$ with $|\xi| \geq 4n$, where

$$C' = \frac{C_0C}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-N(2\eta/h)} d\eta < \infty.$$

The result now follows from Lemma 7.5.2.

(iii) Let $n \in \mathbb{N}$ be arbitrary. We have that

$$\begin{aligned} |1 - \widehat{\rho}_n(\xi)| &= \left| 1 - \frac{r_n^d}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(\eta/n) \widehat{\chi}(r_n(\xi - \eta)) d\eta \right| \\ &= \left| \frac{r_n^d}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - \psi(\eta/n)) \widehat{\chi}(r_n(\xi - \eta)) d\eta \right| \\ &\leq \frac{r_n^d}{(2\pi)^d} \int_{|\eta| \geq n} |\widehat{\chi}(r_n(\xi - \eta))| d\eta \\ &= \frac{1}{(2\pi)^d} \int_{|\xi - \frac{\eta}{r_n}| \geq n} |\widehat{\chi}(\eta)| d\eta. \end{aligned}$$

Lemma 2.2.9 implies that $|\widehat{\chi}(\eta)| \leq Ce^{-N(2H^2\lambda\eta)}$, $\eta \in \mathbb{R}^d$, for some $C > 0$. Furthermore, notice that $|\eta| \geq r_n n/2$ for all $\xi, \eta \in \mathbb{R}^d$ with $|\xi| \leq n/2$ and $|\xi - \eta/r_n| \geq n$. Hence, we obtain that $|1 - \widehat{\rho}_n(\xi)| \leq C'e^{-N(H\lambda r_n)}$ for all $\xi \in \mathbb{R}^d$ with $|\xi| \leq n/2$, where

$$C' = \frac{C_0C}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-N(2H\lambda t)} dt < \infty.$$

The result now follows from Lemma 7.5.2. □

Lemma 7.5.4. $(\Phi_n)_n \in \text{TO}_{\text{loc}}^*(\Omega)$.

Proof. We already observed that $\Phi_n \in L(\mathcal{D}^*(\Omega), \mathcal{E}^*(\Omega))$ for $n \in \mathbb{N}$ fixed. Next, notice that $(\Phi_n)_n$ is localizing because $\lim_{n \rightarrow \infty} r_n = 0$. We now show that $(\Phi_n)_n$ satisfies $(\text{TO})_j$, $j = 1, 2, 3$, with the aid of Lemma 7.5.1. We shall only treat the Beurling case; the Roumieu case is similar.

(TO)₁ Let $f \in \mathcal{D}'^{(M_p)}(\Omega)$, $K \Subset \Omega$, and $h > 0$ be arbitrary. There is $j \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $\text{supp } \rho_n(x - \cdot) \subseteq K_j$ for all $x \in K$ and $n \geq n_0$. Set $\kappa = \kappa_j$. Hence, $\Phi_n(f)(x) = (\kappa f * \rho_n)(x)$ for $x \in K$ and $n \geq n_0$. By Lemma 2.2.7 it suffices to show that

$$\int_{\mathbb{R}^d} |\widehat{\kappa f}(\xi)| |\widehat{\rho}_n(\xi)| e^{M(\xi/h)} d\xi = O(e^{M(\lambda n)})$$

for some $\lambda > 0$. Notice that $|\widehat{\kappa f}(\xi)| \leq C e^{M(\xi/k)}$, $\xi \in \mathbb{R}^d$, for some $C, k > 0$. Lemma 7.5.3(ii) implies that

$$\int_{|\xi| \geq 4n} |\widehat{\kappa f}(\xi)| |\widehat{\rho}_n(\xi)| e^{M(\xi/h)} d\xi = O(e^{M(\lambda n)}).$$

On the other hand, by Lemma 7.5.3(i), we have that

$$\begin{aligned} & \int_{|\xi| \leq 4n} |\widehat{\kappa f}(\xi)| |\widehat{\rho}_n(\xi)| e^{M(\xi/h)} d\xi \\ & \leq \frac{C_0 C \|\widehat{\chi}\|_{L^1}}{(2\pi)^d} \int_{|\xi| \leq 4n} e^{M(\xi/k) + M(H\xi/h) - M(\xi/h)} d\xi \\ & \leq C' e^{M(\lambda n)}, \end{aligned}$$

where $\lambda = 4H \max\{1/k, H/h\}$ and

$$C' = \frac{C_0 C \|\widehat{\chi}\|_{L^1}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-M(\xi/h)} d\xi < \infty.$$

(TO)₂ By the null characterization of the ideal $\mathcal{E}_{\mathcal{N}}^{(M_p)}(\Omega)$ (Proposition 3.4.5) we only need to show that

$$\begin{aligned} & \forall \varphi \in \mathcal{E}^{(M_p)}(\Omega) \forall K \Subset \Omega \forall \lambda > 0 : \\ & \sup_{n \in \mathbb{N}} \max_{x \in K} |\Phi_n(\varphi)(x) - \varphi(x)| e^{M(\lambda n)} < \infty. \end{aligned}$$

There is $j \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $\text{supp } \rho_n(x - \cdot) \subseteq K_j$ for all $x \in K$ and $n \geq n_0$. Set $\kappa = \kappa_j$. Hence, $\Phi_n(\varphi)(x) - \varphi(x) = (\kappa \varphi * \rho_n)(x) - \kappa(x) \varphi(x)$ for $x \in K$ and $n \geq n_0$ and, thus,

$$\begin{aligned} \max_{x \in K} |\Phi_n(\varphi)(x) - \varphi(x)| &= \max_{x \in K} |(\kappa \varphi * \rho_n)(x) - \kappa(x) \varphi(x)| \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{\kappa \varphi}(\xi)| |1 - \widehat{\rho}_n(\xi)| d\xi. \end{aligned}$$

Therefore, it suffices to show that

$$\int_{\mathbb{R}^d} |\widehat{\kappa\varphi}(\xi)| |1 - \widehat{\rho}_n(\xi)| d\xi = O(e^{-M(\lambda n)}).$$

By Lemma 2.2.9 it holds that $|\widehat{\kappa\varphi}(\xi)| \leq Ce^{-M(2H\lambda\xi)}$, $\xi \in \mathbb{R}^d$, for some $C > 0$. Lemma 7.5.3(iii) implies that

$$\int_{|\xi| \leq n/2} |\widehat{\kappa\varphi}(\xi)| |1 - \widehat{\rho}_n(\xi)| d\xi = O(e^{-M(\lambda n)}).$$

On the other hand, by Lemma 7.5.3(i), we have that

$$\int_{|\xi| \geq n/2} |\widehat{\kappa\varphi}(\xi)| |1 - \widehat{\rho}_n(\xi)| d\xi \leq C'e^{-M(\lambda n)},$$

where

$$C' = C_0 C \left(1 + \frac{\|\widehat{\chi}\|_{L^1}}{(2\pi)^d} \right) \int_{\mathbb{R}^d} e^{-M(2\lambda\xi)} d\xi < \infty.$$

(TO)₃ Since the space $\mathcal{D}^*(\Omega)$ is Montel, it suffices to show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \Phi_n(f)(x) \varphi(x) dx = \langle f, \varphi \rangle, \quad \varphi \in \mathcal{D}^*(\Omega),$$

for all $f \in \mathcal{D}'^*(\Omega)$. There is $j \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $\text{supp } \rho_n(x - \cdot) \subseteq K_j$ for all $x \in \text{supp } \varphi$ and $n \geq n_0$. Hence, for $\kappa = \kappa_j$, we have that

$$\begin{aligned} \int_{\mathbb{R}^d} \Phi_n(f)(x) \varphi(x) dx &= \int_{\mathbb{R}^d} \langle f(y), \kappa(y) \rho_n(x - y) \rangle \varphi(x) dx \\ &= \langle f(y), \kappa(y) \int_{\mathbb{R}^d} \rho_n(x - y) \varphi(x) dx \rangle \\ &= \langle f(y), \kappa(y) \Phi_n(\varphi)(y) \rangle \end{aligned}$$

for $n \geq n_0$. The result now follows from (TO)₂ and the continuity of f . \square

Lemma 7.5.4 enables us to apply the general theory developed in Sections 7.2-7.4 to the test pair of sheaves $(\mathcal{D}'^*, \mathcal{E}^*)$. In particular, we obtain that there exists a fine sheaf $\mathcal{G}_{\text{loc}}^*$ of algebras,

a sheaf monomorphism $\iota : \mathcal{D}'^* \rightarrow \mathcal{G}_{\text{loc}}^*$, and a sheaf monomorphism of algebras $\sigma : \mathcal{E}^* \rightarrow \mathcal{G}_{\text{loc}}^*$ such that $\iota|_{\mathcal{E}^*} = \sigma$. Next, since the partial derivatives ∂_i , $i = 1, \dots, d$, satisfy the assumptions of Lemma 7.4.17, there are sheaf morphisms $\widehat{\partial}_i : \mathcal{G}_{\text{loc}}^* \rightarrow \mathcal{G}_{\text{loc}}^*$ such that $\widehat{\partial}_i \circ \iota = \iota \circ \partial_i$. Moreover, for each open set $\Omega \subseteq \mathbb{R}^d$, the mapping $\widehat{\partial}_i : \mathcal{G}_{\text{loc}}^*(\Omega) \rightarrow \mathcal{G}_{\text{loc}}^*(\Omega)$ is a derivation. Finally, as shown in [91, p. 626], every real analytic diffeomorphism $\mu : \Omega' \rightarrow \Omega$ induces an isomorphism $\mathcal{D}'^*(\Omega) \rightarrow \mathcal{D}'^*(\Omega')$ given by

$$\langle \mu^*(f), \varphi \rangle := \left\langle f(x), \frac{\varphi(\mu^{-1}(x))}{|J_\mu(\mu^{-1}(x))|} \right\rangle, \quad \varphi \in \mathcal{D}^*(\Omega'),$$

where $J_\mu = \det d\mu$ is the Jacobian of μ . The restriction of μ^* to $\mathcal{E}^*(\Omega)$ coincides with the classical pullback $\mathcal{E}^*(\Omega) \rightarrow \mathcal{E}^*(\Omega') : \varphi \rightarrow \varphi \circ \mu$. Hence, by Lemma 7.3.5, we obtain a corresponding isomorphism $\bar{\mu}^* : \mathcal{G}_{\text{loc}}^*(\Omega) \rightarrow \mathcal{G}_{\text{loc}}^*(\Omega')$ such that $\bar{\mu}^* \circ \iota_\Omega = \iota_{\Omega'} \circ \mu^*$. Observe that the assignment $\mu \rightarrow \bar{\mu}^*$ is contravariant functorial, i.e. if $\mu : \Omega' \rightarrow \Omega$ and $\nu : \Omega'' \rightarrow \Omega'$ are real analytic diffeomorphisms and we set $\tau = \mu \circ \nu$, then $\bar{\tau}^* = \bar{\nu}^* \circ \bar{\mu}^*$. Summarizing, we have shown the following result.

Theorem 7.5.5. *There exist a fine sheaf $\mathcal{G}_{\text{loc}}^*$ of algebras, a sheaf monomorphism $\iota : \mathcal{D}'^* \rightarrow \mathcal{G}_{\text{loc}}^*$, and a sheaf monomorphism of algebras $\sigma : \mathcal{E}^* \rightarrow \mathcal{G}_{\text{loc}}^*$ such that the following properties hold:*

- (i) *The restriction of ι to \mathcal{E}^* coincides with σ .*
- (ii) *There are sheaf morphisms $\widehat{\partial}_i : \mathcal{G}_{\text{loc}}^* \rightarrow \mathcal{G}_{\text{loc}}^*$, $i = 1, \dots, d$, of derivations such that $\widehat{\partial}_i \circ \iota = \iota \circ \partial_i$.*
- (iii) *If $\mu : \Omega' \rightarrow \Omega$ is a real analytic diffeomorphism, then there is an isomorphism $\bar{\mu}^* : \mathcal{G}_{\text{loc}}^*(\Omega) \rightarrow \mathcal{G}_{\text{loc}}^*(\Omega')$ such that $\bar{\mu}^* \circ \iota_\Omega = \iota_{\Omega'} \circ \mu^*$. Moreover, the assignment $\mu \rightarrow \bar{\mu}^*$ is contravariant functorial.*

Part II

Topological properties of convolutor spaces

Chapter 8

Introduction

In his fundamental book [151], Schwartz introduced the space of rapidly decreasing distributions $\mathcal{O}'_C(\mathbb{R}^d)$ and showed that it is in fact equal to the space of convolutors of $\mathcal{S}(\mathbb{R}^d)$, namely, a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $\mathcal{O}'_C(\mathbb{R}^d)$ if and only if $f * \varphi \in \mathcal{S}(\mathbb{R}^d)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ [151, Thm. IX, p. 244]. This characterization suggests to endow $\mathcal{O}'_C(\mathbb{R}^d)$ with the initial topology with respect to the mapping

$$\mathcal{O}'_C(\mathbb{R}^d) \rightarrow L_b(\mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi)^1.$$

This definition entails that $\mathcal{O}'_C(\mathbb{R}^d)$ is semi-reflexive and nuclear. A detailed study of the locally convex structure of $\mathcal{O}'_C(\mathbb{R}^d)$ was carried out by Grothendieck in the last part of his doctoral thesis [73]. He showed that this space is ultrabornological [73, Chap. II, Thm. 16, p. 131] and that its strong dual is isomorphic to the (LF) -space $\mathcal{O}_C(\mathbb{R}^d)$ of slowly increasing smooth functions [73, Chap. II, p. 131]. Hence, $\mathcal{O}_C(\mathbb{R}^d)$ is complete and its strong dual is isomorphic to $\mathcal{O}'_C(\mathbb{R}^d)$. We refer to [105, 5, 106, 125, 6] for modern works concerning these spaces.

¹The continuity of the mapping $\varphi \rightarrow f * \varphi$ follows from the closed graph theorem and the continuity of the mapping $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d) : \varphi \rightarrow f * \varphi$.

In this part, we shall investigate similar questions for convolutor spaces of Gelfand-Shilov spaces [64, 63], i.e. spaces consisting of smooth or ultradifferentiable functions satisfying global decay estimates with respect to a general sequence of weights. The main goal is to determine a topological predual of these spaces and to characterize when they are ultrabornological in terms of their defining sequence of weights. The smooth case is studied in Chapter 10. In fact, we shall consider there a general class of weighted L^1 convolutor spaces. Consequently, our results will simultaneously apply to convolutor spaces of the Gelfand-Shilov spaces $\mathcal{K}\{M_p\}$ of smooth functions² [63] and to weighted versions of the space \mathcal{D}'_{L_1} of integrable distributions [151]. Various classical results of Schwartz concerning \mathcal{D}'_{L_1} [151, pp. 201-203] will be extended to the weighted setting; see [51] for earlier work in this direction. The space of convolutors of the space $\mathcal{K}_1(\mathbb{R}^d)$ of exponentially rapidly decreasing smooth functions [75] was studied by Zielezny [166]. He claims that this space is ultrabornological but his argument seems to contain a gap (see Remark 10.4.10); Theorem 10.4.7 contains this result as a particular instance. In Chapter 11, we treat the ultradifferentiable case. Spaces of convolutors of the Gelfand-Shilov spaces $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ were studied in [52, 50]. The authors of [50] constructed test function spaces whose duals coincide algebraically with these convolutor spaces and asked whether these equalities also hold topologically [50, p. 413]. We shall investigate this problem from a broader perspective and solve it at the end of this chapter.

²In this notation, $(M_p)_{p \in \mathbb{N}}$ stands for an increasing sequence of positive continuous weight functions on \mathbb{R}^d and *not* for a weight sequence. More precisely, $\mathcal{K}\{M_p\}$ is defined as the Fréchet space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\max_{|\alpha| \leq p} \sup_{x \in \mathbb{R}^d} |\partial^\alpha \varphi(x)| M_p(x) < \infty$$

for all $p \in \mathbb{N}$. In order not to cause any confusion, we shall adopt a different notation in Chapter 10.

It is important to point out that the methods to be employed are completely different from the ones used by Grothendieck and the authors of [50] (who followed Schwartz's approach via parametrices [151]). Namely, we first introduce weighted (LF) -spaces of smooth and ultradifferentiable functions and study their completeness with the aid of the abstract results presented in Subsection 2.3.1; these spaces are the natural analogue of $\mathcal{O}_C(\mathbb{R}^d)$ in the present setting. Next, we establish the mapping properties of the short-time Fourier transform (STFT) [68] on various function and (ultra)distribution spaces. Finally, we exploit these mapping properties to show that the duals of the aforementioned (LF) -spaces coincide algebraically with the convolutor spaces we are interested in and to link the topological properties of these spaces with those of the (LF) -spaces. We strongly believe that this method, especially the use of the STFT, leads to transparent proofs of rather subtle results. In this regard, we highlight the papers [6] in which the mapping properties of the STFT on $\mathcal{O}'_C(\mathbb{R}^d)$ are established by using Schwartz's theory of vector-valued distributions and [96] in which weighted \mathcal{B}' and $\dot{\mathcal{B}}'$ spaces are characterized in terms of the growth of convolution averages of their elements via the STFT. We were inspired by both of these works.

Chapter 9

Preliminaries

In the first part of this chapter, we state various important results from the theory of weighted inductive limits of spaces of continuous functions [13, 10, 7, 4, 11]. They will be frequently used in the sequel. Next, we introduce the Gelfand-Shilov spaces $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ [64, 134] and collect some basic facts about them. Finally, we define and study the short-time Fourier transform [68] on the space $\mathcal{D}'(\mathbb{R}^d)$ of distributions and on the tempered ultradistribution spaces $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$. To this end, we shall closely follow the paper [96], where the short-time Fourier transform is defined on the space of distributions of exponential type.

9.1 Weighted inductive limits of spaces of continuous functions

In this section, we discuss the regularity properties and the related problem of projective description of weighted inductive limits of spaces of continuous functions. This subject has a long tradition and goes back to the pioneer work of Bierstedt, Meise, and Summers [12, 13]. We shall employ results about the (LB) -case, the vector-valued case, as well as the (LF) -case in Chapters 10 and 11.

9.1.1 The (LB) -case

Let X be a locally compact Hausdorff topological space. For a non-negative function v on X we write $Cv(X)$ for the seminormed space consisting of all $f \in C(X)$ such that $\|f\|_{Cv} := \sup_{x \in X} |f(x)|v(x) < \infty$. If v is positive, then $\|\cdot\|_{Cv}$ is actually a norm and if, in addition, $1/v$ is locally bounded, then $Cv(X)$ is complete and, thus, a Banach space. These requirements are fulfilled if v is positive and continuous. We denote by $C(v)_0(X)$ the closed subspace of $Cv(X)$ consisting of functions f such that fv vanishes at ∞ .

A (pointwise) decreasing sequence $\mathcal{V} = (v_N)_{N \in \mathbb{N}}$ of positive continuous functions on X is called a *decreasing weight system on X* . We define the (LB) -spaces

$$\mathcal{V}C(X) := \varinjlim_{N \in \mathbb{N}} Cv_N(X), \quad \mathcal{V}_0C(X) := \varinjlim_{N \in \mathbb{N}} C(v_N)_0(X).$$

We shall sometimes use the following condition on \mathcal{V} (cf. condition (S) from [10]):

$$\forall N \exists M > N : v_M/v_N \text{ vanishes at } \infty. \quad (9.1.1)$$

In such a case, $\mathcal{V}C(X) = \mathcal{V}_0C(X)$. We start with the following result.

Proposition 9.1.1. ([13, Cor. 2.7]) *Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system on X . Then, $\mathcal{V}C(X)$ is boundedly retractive if and only if \mathcal{V} is regularly decreasing, i.e.*

$$\begin{aligned} \forall N \exists M \geq N \forall \varepsilon > 0 \forall K \geq M \exists \delta > 0 \forall x \in X : \\ \varepsilon v_N(x) \leq v_M(x) \implies \delta v_N(x) \leq v_K(x). \end{aligned}$$

Next, we discuss the problem of projective description for the spaces $\mathcal{V}C(X)$ [13, 10, 7]. The *maximal Nachbin family associated with \mathcal{V}* is given by the space $\bar{V} = \bar{V}(\mathcal{V})$ consisting of all non-negative upper semicontinuous functions v on X such that $\sup_{x \in X} v(x)/v_N(x) < \infty$ for all $N \in \mathbb{N}$. The following lemma is well-known.

Lemma 9.1.2. *Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system on X and let $f : X \rightarrow [0, \infty)$. Then, $\sup_{x \in X} f(x)v_N(x) < \infty$ for some $N \in \mathbb{N}$ if and only if $\sup_{x \in X} f(x)v(x) < \infty$ for all $v \in \overline{V}$.*

The *projective hull* of $\mathcal{V}C(X)$, denoted by $C\overline{V}(X)$, is defined as the space consisting of all $f \in C(X)$ such that $\|f\|_{Cv} < \infty$ for all $v \in \overline{V}$. The space $C\overline{V}(X)$ is endowed with the locally convex topology generated by the system of seminorms $\{\|\cdot\|_{Cv} \mid v \in \overline{V}\}$. Lemma 9.1.2 implies that $\mathcal{V}C(X)$ and $C\overline{V}(X)$ coincide algebraically and that these spaces have the same bounded sets. Consequently, $\mathcal{V}C(X)$ is always regular. The problem of projective description is to characterize the weight systems \mathcal{V} for which the spaces $\mathcal{V}C(X)$ and $C\overline{V}(X)$ coincide topologically. In this regard, there is the following important result due to Bastin.

Theorem 9.1.3 ([7, p. 396]). *Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system on X satisfying condition (V), i.e. for every sequence of positive numbers $(\lambda_N)_N$ there is $v \in \overline{V}$ such that for every $N \in \mathbb{N}$ there is $M \in \mathbb{N}$ such that $\min\{\lambda_1 v_1, \dots, \lambda_M v_M\} \leq \max\{v_N/N, v\}$. Then, $\mathcal{V}C(X)$ and $C\overline{V}(X)$ coincide topologically.*

Remark 9.1.4. Bastin also showed that, if for every $v \in \overline{V}$ there is a positive continuous function $\bar{v} \in \overline{V}$ such that $v \leq \bar{v}$, condition (V) is also necessary for the topological identity $\mathcal{V}C(X) = C\overline{V}(X)$. If X is a discrete or a locally compact σ -compact Hausdorff topological space, then every decreasing weight system \mathcal{V} on X satisfies the above condition [13, p. 112].

Remark 9.1.5. We have that

$$(9.1.1) \implies \text{regularly decreasing} \implies (V).$$

On the other hand, every constant weight system is regularly decreasing but obviously does not satisfy (9.1.1).

We now briefly discuss the tensor product of decreasing weight systems. Let X and Y be locally compact Hausdorff spaces and let

$\mathcal{V}_1 = (v_{1,N})_N$ and $\mathcal{V}_2 = (v_{2,N})_N$ be decreasing weight systems on X and Y , respectively. We write $\mathcal{V}_1 \otimes \mathcal{V}_2 = (v_{1,N} \otimes v_{2,N})_N$ for the decreasing weight system on $X \times Y$ given by $v_{1,N} \otimes v_{2,N}(x, y) = v_{1,N}(x)v_{2,N}(y)$, $x \in X, y \in Y$. We shall need the following two technical lemmas whose verification is left to the reader.

Lemma 9.1.6. *Let $\mathcal{V}_1 = (v_{1,N})_N$ and $\mathcal{V}_2 = (v_{2,N})_N$ be decreasing weight systems on X and Y , respectively. Then, for every $v \in \overline{V}(\mathcal{V}_1 \otimes \mathcal{V}_2)$ there are $v_1 \in \overline{V}(\mathcal{V}_1)$ and $v_2 \in \overline{V}(\mathcal{V}_2)$ such that $v \leq v_1 \otimes v_2$.*

Lemma 9.1.7. *Let $\mathcal{V}_1 = (v_{1,N})_N$ and $\mathcal{V}_2 = (v_{2,N})_N$ be decreasing weight systems on X and Y , respectively. If both \mathcal{V}_1 and \mathcal{V}_2 satisfy (V) , then also $\mathcal{V}_1 \otimes \mathcal{V}_2$ satisfies (V) .*

An increasing sequence $\mathcal{W} = (w_n)_{n \in \mathbb{N}}$ of positive continuous functions on X is called an *increasing weight system on X* . We define the ensuing Fréchet spaces

$$\mathcal{WC}(X) := \varprojlim_{n \in \mathbb{N}} Cw_n(X), \quad \mathcal{W}_0C(X) := \varprojlim_{n \in \mathbb{N}} C(w_n)_0(X).$$

Similarly to (9.1.1), we shall sometimes use the following condition on \mathcal{W} (cf. condition (P) from [64]):

$$\forall n \exists m > n : w_n/w_m \text{ vanishes at } \infty. \quad (9.1.2)$$

In such a case, $\mathcal{WC}(X) = \mathcal{W}_0C(X)$.

Let X and Y be locally compact Hausdorff spaces and let $\mathcal{W}_1 = (w_{1,n})_n$ and $\mathcal{W}_2 = (w_{2,n})_n$ be decreasing weight systems on X and Y , respectively. We write $\mathcal{W}_1 \otimes \mathcal{W}_2 = (w_{1,n} \otimes w_{2,n})_n$, an increasing weight system on $X \times Y$.

For a decreasing weight system $\mathcal{V} = (v_N)_N$ on X we define its dual increasing weight system as $\mathcal{V}^\circ = (1/v_N)_N$. Likewise, for an increasing weight system $\mathcal{W} = (w_n)_n$ on X we define its dual decreasing weight system as $\mathcal{W}^\circ = (1/w_N)_N$.

We end this subsection by introducing two conditions that will play a major role in the sequel. A decreasing weight system $\mathcal{V} = (v_N)_N$ on X is said to satisfy condition (Ω) if

$$\forall N \exists M \geq N \forall K \geq M \exists \theta \in (0, 1) \exists C > 0 \forall x \in X : \\ v_M(x) \leq C v_N(x)^{1-\theta} v_K(x)^\theta.$$

An increasing weight system $\mathcal{W} = (w_n)_n$ on X is said to satisfy condition (DN) if

$$\exists n \forall m \geq n \exists k \geq m \exists C > 0 \forall x \in X : w_m(x)^2 \leq C w_n(x) w_k(x).$$

Remark 9.1.8. Conditions (Ω) and (DN) are inspired by and closely connected with Vogt's topological invariants (Ω) and (DN) for Fréchet spaces [114]. These conditions play an essential role in the splitting theory for Fréchet spaces [162, 158] and, in fact, the ideas of some of our proofs in Sections 10.2 and 11.2 stem from this theory.

9.1.2 The vector-valued case

Let X be a locally compact Hausdorff topological space and let $\mathcal{V} = (v_N)_N$ be a decreasing weight system on X . Let E be a Fréchet space with a fundamental sequence of seminorms $(\|\cdot\|_n)_n$. For each $N \in \mathbb{N}$ we write $Cv_N(X; E)$ for the Fréchet space consisting of all $f \in C(X; E)$ such that

$$\sup_{x \in X} \|f(x)\|_n v_N(x) < \infty$$

for all $n \in \mathbb{N}$. We define the following (LF) -space

$$\mathcal{VC}(X; E) := \varinjlim_{N \in \mathbb{N}} Cv_N(X; E).$$

Albanese showed that, if E is non-normable, all the regularity conditions considered in Subsection 2.3.1 are equivalent for the space $\mathcal{VC}(X; E)$ and characterized them in the ensuing way.

Theorem 9.1.9 ([4, Thm. 2.3]). *Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system on X and let E be a non-normable Fréchet space with a fundamental increasing sequence of seminorms $(\|\cdot\|_n)_n$. Then, the following statements are equivalent:*

- (i) $\mathcal{VC}(X; E)$ is boundedly retractive.
- (ii) $\mathcal{VC}(X; E)$ satisfies (wQ) .
- (iii) The pair (E, \mathcal{V}) satisfies condition $(S_2)^*$, i.e.

$$\forall N \exists M \geq N \exists n \forall K \geq M \forall m \exists k \exists C > 0 \forall e \in E \forall x \in X : \\ v_M(x) \|e\|_m \leq C(v_N(x) \|e\|_n + v_K(x) \|e\|_k).$$

Remark 9.1.10. Let \mathcal{V} be a decreasing weight system on X and let E and F be Fréchet spaces. If E and F are topologically isomorphic, then (E, \mathcal{V}) satisfies $(S_2)^*$ if and only if (F, \mathcal{V}) does so.

Remark 9.1.11. Condition $(S_2)^*$ is inspired by and closely connected with the condition (S_2^*) for two general Fréchet spaces [158]. This conditions play an important role in the splitting theory for Fréchet spaces [162, 158].

We now further analyze the condition $(S_2)^*$. Since this is very similar to the analysis of the conditions (S_1^*) and (S_2^*) in the splitting theory for Fréchet spaces [158], we omit all proofs. A Fréchet space E with a fundamental increasing sequence of seminorms $(\|\cdot\|_n)_n$ is said to satisfy condition (DN) [114, p. 359] if

$$\exists n \forall m \geq n \exists k \geq m \exists C > 0 \forall e \in E : \|e\|_m^2 \leq C \|e\|_n \|e\|_k.$$

Lemma 9.1.12 (cf. [158, Thm. 5.1]). *Let \mathcal{V} be a decreasing weight system on X and let E be a Fréchet space. If \mathcal{V} satisfies (Ω) and E satisfies (DN) , then (E, \mathcal{V}) satisfies $(S_2)^*$.*

Next, we consider the case that E is a power series space. Let $\beta = (\beta_j)_{j \in \mathbb{N}}$ be a strictly increasing positive real sequence such that

$\beta_j \rightarrow \infty$. The sequence β is said to be *shift-stable* if $\sup_j \beta_{j+1}/\beta_j < \infty$. As customary [114, Chap. 29], we denote by $\Lambda_\infty(\beta)$ the Fréchet space consisting of all sequences $(x_j)_j \in \mathbb{C}^\mathbb{N}$ such that

$$\left(\sum_{j=0}^{\infty} |x_j|^2 e^{2n\beta_j} \right)^{1/2} < \infty$$

for all $n \in \mathbb{N}$, and by $\Lambda_0(\beta)$ the Fréchet space consisting of all sequences $(x_j)_j \in \mathbb{C}^\mathbb{N}$ such that

$$\left(\sum_{j=0}^{\infty} |x_j|^2 e^{-2\beta_j/n} \right)^{1/2} < \infty$$

for all $n \in \mathbb{N}$. We then have:

Lemma 9.1.13 (cf. [158, Thm. 4.1]). *Let \mathcal{V} be a decreasing weight system on X and let β be shift-stable. Then, \mathcal{V} satisfies (Ω) if and only if $(\Lambda_\infty(\beta), \mathcal{V})$ satisfies $(S_2)^*$.*

Finally, we introduce an analogue of $(S_2)^*$ for a pair consisting of an increasing weight system on X and an (LB) -space. Let $\mathcal{W} = (w_n)_n$ be an increasing weight system on X and let $E = \varinjlim_N E_N$ be an (LB) -space. The pair (\mathcal{W}, E) is said to satisfy condition $(S_2)^*$ if

$$\forall N \exists M \geq N \exists n \forall K \geq M \forall m \exists k \exists C > 0 \forall e \in E_N \forall x \in X :$$

$$\|e\|_{E_M} w_m(x) \leq C(\|e\|_{E_N} w_n(x) + \|e\|_{E_K} w_k(x)).$$

Remark 9.1.14. Let \mathcal{W} be an increasing weight system on X and let E and F be (LB) -spaces. If E and F are topologically isomorphic, then (\mathcal{W}, E) satisfies $(S_2)^*$ if and only if (\mathcal{W}, F) does so, as follows from Grothendieck's factorization theorem.

We have the ensuing analogue of Lemma 9.1.13.

Lemma 9.1.15 (cf. [158, Thm. 4.3]). *Let \mathcal{W} be an increasing weight system on X and let β be shift-stable. Then, \mathcal{W} satisfies (DN) if and only if $(\mathcal{W}, \Lambda'_0(\beta))$ satisfies $(S_2)^*$.*

9.1.3 The (LF) -case

Let X be a locally compact Hausdorff topological space. A double sequence $\mathcal{U} := (u_{N,n})_{(N,n) \in \mathbb{N}^2}$ of positive continuous functions on X is called a *general weight system on X* if $u_{N,n} \geq u_{N+1,n}$ and $u_{N,n} \leq u_{N,n+1}$ for all $N, n \in \mathbb{N}$. We define the following (LF) -spaces

$$\mathcal{UC}(X) := \varinjlim_{N \in \mathbb{N}} \varprojlim_{n \in \mathbb{N}} C u_{N,n}(X), \quad \mathcal{U}_0 C(X) := \varinjlim_{N \in \mathbb{N}} \varprojlim_{n \in \mathbb{N}} C(u_{N,n})_0(X).$$

In this subsection, we discuss the problem of projective description for the spaces $\mathcal{UC}(X)$. The *maximal Nachbin family associated with \mathcal{U}* is given by the space $\overline{\mathcal{U}} = \overline{\mathcal{U}}(\mathcal{U})$ consisting of all non-negative upper semicontinuous functions u on X such that for all $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\sup_{x \in X} u(x)/u_{N,n}(x) < \infty$. The *projective hull of $\mathcal{UC}(X)$* , denoted by $C\overline{\mathcal{U}}(X)$, is defined as the space consisting of all $f \in C(X)$ such that $\|f\|_{C u} < \infty$ for all $u \in \overline{\mathcal{U}}$. The space $C\overline{\mathcal{U}}(X)$ is endowed with the locally convex topology generated by the system of seminorms $\{\|\cdot\|_{C u} \mid u \in \overline{\mathcal{U}}\}$. The problem of projective description in this setting is to find conditions on \mathcal{U} which ensure that $\mathcal{UC}(X)$ and $C\overline{\mathcal{U}}(X)$ coincide algebraically and/or topologically. This problem was thoroughly studied by Bierstedt and Bonet in [11]. We shall use the following result of these authors.

Proposition 9.1.16. ([11, Thm. 3, p. 36 and Cor. 5, p. 42]) *Let \mathcal{U} be a general weight system on X . If $\mathcal{UC}(X)$ is boundedly retractive, then the spaces $\mathcal{UC}(X)$ and $C\overline{\mathcal{U}}(X)$ coincide algebraically and topologically.*

We now further specialize Proposition 9.1.16 to general weight systems arising as the tensor product of a decreasing and an increasing weight system. More precisely, let X and Y be locally compact Hausdorff spaces, let $\mathcal{V} = (v_N)_N$ be a decreasing weight system on X and let $\mathcal{W} = (w_n)_n$ be an increasing weight system on Y . We define $\mathcal{V} \otimes_i \mathcal{W} := (v_N \otimes w_n)_{N,n}$, a general weight system on $X \times Y$.

Corollary 9.1.17. *Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system on X and let $\mathcal{W} = (w_n)_n$ be an increasing weight system on Y . Set $U = \mathcal{V} \otimes_i \mathcal{W}$. If \mathcal{V} satisfies (Ω) and \mathcal{W} satisfies (DN) , then $\mathcal{UC}(X \times Y)$ is boundedly retractive. Consequently, the spaces $\mathcal{UC}(X \times Y)$ and $C\overline{U}(X \times Y)$ coincide algebraically and topologically in such a case.*

Proof. Observe that $\mathcal{UC}(X \times Y) \cong \mathcal{VC}(X; \mathcal{WC}(Y))$ topologically. By Proposition 9.1.9 it therefore suffices to show that the pair $(\mathcal{V}, \mathcal{WC}(Y))$ satisfies $(S_2)^*$ but this follows directly from 9.1.12. \square

9.2 The Gelfand-Shilov spaces $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$

Let M_p and A_p be two weight sequences. In this section, we introduce and briefly discuss the Gelfand-Shilov spaces $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$.

We denote by A the associated function of A_p . For $h, \lambda > 0$ we write $\mathcal{S}_{A_p, \lambda}^{M_p, h}(\mathbb{R}^d)$ for the Banach space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{S}_{A_p, \lambda}^{M_p, h}} := \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)| e^{A(x/\lambda)}}{h^{|\alpha|} M_\alpha} < \infty.$$

We define

$$\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) := \varprojlim_{h \rightarrow 0^+} \mathcal{S}_{A_p, h}^{M_p, h}(\mathbb{R}^d), \quad \mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d) := \varinjlim_{h \rightarrow \infty} \mathcal{S}_{A_p, h}^{M_p, h}(\mathbb{R}^d).$$

The spaces $\mathcal{S}_{\{p!^\sigma\}}^{\{p!^\tau\}}(\mathbb{R}^d) = \mathcal{S}_\tau^\sigma(\mathbb{R}^d)$, $\sigma, \tau > 0$, are the classical Gelfand-Shilov spaces introduced in [64].

Elements of the dual spaces $\mathcal{S}'_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{S}'_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ are called *tempered ultradistributions of class (M_p) and type (A_p) (of Beurling type)* and *tempered ultradistributions of class $\{M_p\}$ and type $\{A_p\}$ (of Roumieu type)*, respectively. We shall write $*$ instead of (M_p) or $\{M_p\}$ and \dagger instead of (A_p) or $\{A_p\}$ if we want to treat the Beurling

and Roumieu case simultaneously. We introduce the following set of conditions on M_p and A_p :

$$M_p \text{ and } A_p \text{ satisfy (M.1) and (M.2),} \quad (9.2.1)$$

$$p! \prec M_p A_p, \text{ and } \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \neq \{0\}.$$

Remark 9.2.1. It is a classical result of Gelfand and Shilov that the space $\mathcal{S}_\tau^\sigma(\mathbb{R}^d)$ is non-trivial if and only if $\sigma + \tau \geq 1$ [64, p. 235]. Hence, a sufficient condition for the non-triviality of $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ is $p!^\sigma \subset M_p$ and $p!^\tau \subset A_p$ for some $\sigma, \tau > 0$ with $\sigma + \tau > 1$. In Part III, we will characterize the non-triviality of the spaces $\mathcal{S}_{(A_p)}^{(p!)}(\mathbb{R}^d)$ and $\mathcal{S}_{\{A_p\}}^{\{p!\}}(\mathbb{R}^d)$ in terms of the growth of the weight sequence A_p .

Under (9.2), the spaces $\mathcal{S}_\dagger^*(\mathbb{R}^d)$ and $\mathcal{S}_\dagger'^*(\mathbb{R}^d)$ satisfy the ensuing properties:

- Let $f \in \mathcal{S}_\dagger'^*(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}_\dagger^*(\mathbb{R}^d)$. Then, their convolution given by

$$f * \varphi = \langle f(x), \varphi(\cdot - x) \rangle$$

belongs to $\mathcal{E}^*(\mathbb{R}^d)$ and satisfies $\sup_{x \in \mathbb{R}^d} |f * \varphi(x)| e^{-A(x/\lambda)} < \infty$ for some $\lambda > 0$ (for all $\lambda > 0$).

- The Fourier transform is an isomorphism from $\mathcal{S}_\dagger^*(\mathbb{R}^d)$ onto $\mathcal{S}_*^\dagger(\mathbb{R}^d)$.
- $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ is densely and continuously included in $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ [134, Lemma 2.4].
- $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ is an (FN) -space and $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ is a (DFN) -space [134, Prop. 2.11].

9.3 The short-time Fourier transform

The aim of this section is to define and study the short-time Fourier transform of elements of $\mathcal{D}'(\mathbb{R}^d)$ with respect to compactly supported smooth window functions and of elements of $\mathcal{S}_\dagger'^*(\mathbb{R}^d)$ with

respect to window functions belonging to $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. In particular, we prove reconstruction and desingularization formulas. We start by stating some fundamental properties of the short-time Fourier transform on the space $L^2(\mathbb{R}^d)$.

The translation and modulation operators are denoted by $T_x f = f(\cdot - x)$ and $M_\xi f(t) = e^{2\pi i t \xi} f(t)$, for $x, \xi \in \mathbb{R}^d$. We also write $\check{f} = f(-\cdot)$ for reflection about the origin. The *short-time Fourier transform* (STFT) of a function $f \in L^2(\mathbb{R}^d)$ with respect to a window function $\psi \in L^2(\mathbb{R}^d)$ is defined as

$$\begin{aligned} V_\psi f(x, \xi) &:= (f, M_\xi T_x \psi)_{L^2} \\ &= \int_{\mathbb{R}^d} f(t) \overline{\psi(t-x)} e^{-2\pi i t \xi} dt, \quad (x, \xi) \in \mathbb{R}^{2d}. \end{aligned}$$

We have that $\|V_\psi f\|_{L^2(\mathbb{R}^{2d})} = \|\psi\|_{L^2} \|f\|_{L^2}$. In particular, the mapping $V_\psi : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^{2d})$ is continuous. The adjoint of V_ψ is given by the weak integral

$$V_\psi^* F = \int \int_{\mathbb{R}^{2d}} F(x, \xi) M_\xi T_x \psi dx d\xi, \quad F \in L^2(\mathbb{R}^{2d}).$$

If $\psi \neq 0$ and $\gamma \in L^2(\mathbb{R}^d)$ is a synthesis window for ψ , that is, $(\gamma, \psi)_{L^2} \neq 0$, then

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \text{id}_{L^2(\mathbb{R}^d)}. \quad (9.3.1)$$

For further properties of the STFT we refer to the book [68].

9.3.1 The STFT on $\mathcal{D}'(\mathbb{R}^d)$

In order to be able to extend the STFT to the space $\mathcal{D}'(\mathbb{R}^d)$, we must first establish the mapping properties of the STFT on $\mathcal{D}(\mathbb{R}^d)$. We need some preparation. Let X and Y be locally convex spaces. We write $X \widehat{\otimes}_\pi Y$, $X \widehat{\otimes}_\varepsilon Y$, and $X \widehat{\otimes}_i Y$ for the completion of the tensor product $X \otimes Y$ with respect to the projective topology, the ε -topology, and the inductive topology, respectively. As before,

we simply write $X \widehat{\otimes} Y = X \widehat{\otimes}_\pi Y = X \widehat{\otimes}_\varepsilon Y$ if either X or Y is nuclear. For $K \Subset \mathbb{R}^d$ we define \mathcal{D}_K as the closed subspace of $C^\infty(\mathbb{R}^d)$ consisting of all functions φ with $\text{supp } \varphi \subseteq K$. Let $d_1, d_2 \in \mathbb{N}$ and let $K \Subset \mathbb{R}^{d_1}$. We may identify $\mathcal{D}_{K,x} \widehat{\otimes} \mathcal{S}(\mathbb{R}_\xi^{d_2})$ with the Fréchet space consisting of all $\varphi \in C^\infty(\mathbb{R}_{x,\xi}^{d_1+d_2})$ such that $\text{supp } \varphi \subseteq K \times \mathbb{R}^{d_2}$ and

$$\max_{|\alpha| \leq n} \max_{|\beta| \leq n} \sup_{(x,\xi) \in K \times \mathbb{R}^{d_2}} |\partial_\xi^\beta \partial_x^\alpha \varphi(x, \xi)| (1 + |\xi|)^n < \infty \quad (9.3.2)$$

for all $n \in \mathbb{N}$. By [92, Thm. 2.3] we have the following isomorphism of l.c.s.

$$\mathcal{D}(\mathbb{R}_x^{d_1}) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^{d_2}) \cong \varinjlim_{K \Subset \mathbb{R}^{d_1}} \mathcal{D}_{K,x} \widehat{\otimes} \mathcal{S}(\mathbb{R}_\xi^{d_2}),$$

where the right-hand side is a strict (LF) -space. We then have:

Proposition 9.3.1. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$. Then,*

$$V_\psi : \mathcal{D}(\mathbb{R}^d) \rightarrow \mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d)$$

and

$$V_\psi^* : \mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d) \rightarrow \mathcal{D}(\mathbb{R}^d)$$

are well-defined continuous mappings.

Proof. We start by showing that V_ψ is well-defined and continuous. Consider the continuous linear mappings

$$S : \mathcal{D}(\mathbb{R}_t^d) \rightarrow \mathcal{D}(\mathbb{R}_{x,t}^{2d}) : \varphi(t) \rightarrow \varphi(t) T_x \bar{\psi}(t)$$

and

$$\mathcal{F}_{2\pi,t} : \mathcal{D}(\mathbb{R}_{x,t}^{2d}) \rightarrow \mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d) : \varphi(x, t) \rightarrow \int_{\mathbb{R}^d} \varphi(x, t) e^{-2\pi i t \xi} dt.$$

The result now follows from the representation $V_\psi = \mathcal{F}_{2\pi,t} \circ S$. Next, we treat V_ψ^* is well-defined and continuous. Consider the continuous linear mappings

$$\begin{aligned} S : \mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d) &\rightarrow \mathcal{D}(\mathbb{R}_t^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_{x,\xi}^{2d}) : \\ \varphi(x, \xi) &\rightarrow \varphi(x, \xi) M_\xi T_x \psi(t) \end{aligned}$$

and

$$\begin{aligned} \text{id}_{\mathcal{D}(\mathbb{R}_t^d)} \widehat{\otimes}_i 1(x, \xi) : \mathcal{D}(\mathbb{R}_t^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_{x,\xi}^{2d}) &\rightarrow \mathcal{D}(\mathbb{R}_t^d) : \\ \varphi(t, x, \xi) &\rightarrow \int \int_{\mathbb{R}^{2d}} \varphi(t, x, \xi) dx d\xi. \end{aligned}$$

The result now follows from the representation

$$V_\psi^* = \text{id}_{\mathcal{D}(\mathbb{R}_t^d)} \widehat{\otimes}_i 1(x, \xi) \circ S.$$

□

Observe that, if $\psi \in \mathcal{D}(\mathbb{R}^d) \setminus \{0\}$ and $\gamma \in \mathcal{D}(\mathbb{R}^d)$ is a synthesis window for ψ , the reconstruction formula (9.3.1) reads as

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \text{id}_{\mathcal{D}(\mathbb{R}^d)}. \quad (9.3.3)$$

We are ready to define the STFT on the space $\mathcal{D}'(\mathbb{R}^d)$. For $\psi \in \mathcal{D}(\mathbb{R}^d)$ and $f \in \mathcal{D}'(\mathbb{R}^d)$ we define

$$V_\psi f(x, \xi) := \langle f, \overline{M_\xi T_x \psi} \rangle = e^{-2\pi i x \xi} (f * M_\xi \check{\psi})(x), \quad (x, \xi) \in \mathbb{R}^{2d}.$$

Clearly, $V_\psi f$ is a continuous function on \mathbb{R}^{2d} .

Lemma 9.3.2. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and let $f \in \mathcal{D}'(\mathbb{R}^d)$. Then, for every $K \Subset \mathbb{R}^d$ there is $n \in \mathbb{N}$ such that*

$$\sup_{(x, \xi) \in K \times \mathbb{R}^d} \frac{|V_\psi f(x, \xi)|}{(1 + |\xi|)^n} < \infty.$$

In particular, $V_\psi f$ defines an element of $(\mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d))'$ via

$$\langle V_\psi f, \varphi \rangle := \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) \varphi(x, \xi) dx d\xi, \quad \varphi \in \mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d).$$

Proof. Set $K' = \text{supp } \psi + K$. Since $\text{supp } \overline{M_\xi T_x \psi} \subseteq K'$ for all $(x, \xi) \in K \times \mathbb{R}^d$, there are $n \in \mathbb{N}$ and $C > 0$ such that

$$|V_\psi f(x, \xi)| \leq |\langle f, \overline{M_\xi T_x \psi} \rangle| \leq C \max_{|\alpha| \leq n} \max_{t \in K'} |\partial_t^\alpha (M_\xi T_x \psi)(t)|$$

for all $(x, \xi) \in K \times \mathbb{R}^d$. Observe that

$$\begin{aligned} & \max_{|\alpha| \leq n} \max_{t \in K'} |\partial_t^\alpha (M_\xi T_x \psi)(t)| \\ & \leq \max_{|\alpha| \leq n} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (2\pi |\xi|)^{|\beta|} \max_{t \in K'} |\partial^{\alpha-\beta} \psi(t-x)| \\ & \leq (2\pi)^n \max_{|\alpha| \leq n} \|\partial^\alpha \psi\|_{L^\infty} (1 + |\xi|)^n, \end{aligned}$$

whence the result follows. \square

Lemma 9.3.3. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and let $f \in \mathcal{D}'(\mathbb{R}^d)$. Then,*

$$\langle V_\psi f, \varphi \rangle = \langle f, \overline{V_\psi^* \varphi} \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d).$$

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d)$ be arbitrary. Since

$$\varphi(x, \xi) \overline{M_\xi T_x \psi}(t) \in \mathcal{D}(\mathbb{R}_t^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_{x,\xi}^{2d})$$

and

$$1(x, \xi) \widehat{\otimes}_i f(t) = f(t) \widehat{\otimes}_i 1(x, \xi) \in (\mathcal{D}(\mathbb{R}_t^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_{x,\xi}^{2d}))',$$

we have that

$$\begin{aligned} \langle V_\psi f, \varphi \rangle &= \int \int_{\mathbb{R}^{2d}} \langle f, \overline{M_\xi T_x \psi} \rangle \varphi(x, \xi) dx d\xi \\ &= \langle 1(x, \xi) \widehat{\otimes}_i f(t), \varphi(x, \xi) \overline{M_\xi T_x \psi}(t) \rangle \\ &= \langle f(t) \widehat{\otimes}_i 1(x, \xi), \varphi(x, \xi) \overline{M_\xi T_x \psi}(t) \rangle \\ &= \langle f(t), \int \int_{\mathbb{R}^{2d}} \varphi(x, \xi) \overline{M_\xi T_x \psi}(t) dx d\xi \rangle \\ &= \langle f, \overline{V_\psi^* \varphi} \rangle. \end{aligned}$$

\square

Let $\psi \in \mathcal{D}(\mathbb{R}^d)$. We define the adjoint STFT of an element $F \in (\mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d))'$ as

$$\langle V_\psi^* F, \varphi \rangle := \langle F, \overline{V_\psi \varphi} \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Notice that $V_\psi^* F \in \mathcal{D}'(\mathbb{R}^d)$ because of Proposition 9.3.1. We now have all the necessary ingredients to establish the mapping properties of the STFT on $\mathcal{D}'(\mathbb{R}^d)$.

Proposition 9.3.4. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$. Then,*

$$V_\psi : \mathcal{D}'(\mathbb{R}^d) \rightarrow (\mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d))'$$

and

$$V_\psi^* : (\mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d))' \rightarrow \mathcal{D}'(\mathbb{R}^d)$$

are well-defined continuous mappings. Moreover, if $\psi \neq 0$ and $\gamma \in \mathcal{D}(\mathbb{R}^d)$ is a synthesis window for ψ , then

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \text{id}_{\mathcal{D}'(\mathbb{R}^d)} \quad (9.3.4)$$

and the desingularization formula

$$\langle f, \varphi \rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\overline{\gamma}} \varphi(x, -\xi) dx d\xi \quad (9.3.5)$$

holds for all $f \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Proof. The mapping V_ψ is continuous because of Lemma 9.3.3 and the continuity of $V_\psi^* : \mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d) \rightarrow \mathcal{D}(\mathbb{R}^d)$ (Proposition 9.3.1) while the continuity of V_ψ^* follows directly from the continuity of $V_\psi : \mathcal{D}(\mathbb{R}^d) \rightarrow \mathcal{D}(\mathbb{R}_x^d) \widehat{\otimes}_i \mathcal{S}(\mathbb{R}_\xi^d)$ (Proposition 9.3.1). Next, suppose that $\psi \neq 0$. Let $f \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be arbitrary. Lemma 9.3.3 and the reconstruction formula (9.3.3) imply that

$$\langle V_\gamma^*(V_\psi f), \varphi \rangle = \langle V_\psi f, \overline{V_\gamma \varphi} \rangle = \langle f, \overline{V_\psi^*(V_\gamma \varphi)} \rangle = (\gamma, \varphi)_{L^2} \langle f, \varphi \rangle,$$

whence (9.3.4) and (9.3.5) hold. \square

Finally, we show that, for $f \in \mathcal{E}'(\mathbb{R}^d)$, the desingularization formula (9.3.5) holds for all $\varphi \in \mathcal{E}(\mathbb{R}^d)$. We start with a brief discussion of the STFT on the spaces $\mathcal{E}(\mathbb{R}^d)$ and $\mathcal{E}'(\mathbb{R}^d)$. The space $\mathcal{E}(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}(\mathbb{R}_\xi^d)$ may be identified with the Fréchet space consisting of all $\varphi \in C^\infty(\mathbb{R}_{x,\xi}^{2d})$ such that (9.3.2) holds for all $K \in \mathbb{R}^d$ and $n \in \mathbb{N}$.

Proposition 9.3.5. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$.*

(i) *The mapping*

$$V_\psi : \mathcal{E}(\mathbb{R}^d) \rightarrow \mathcal{E}(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}(\mathbb{R}_\xi^d)$$

is well-defined and continuous.

(ii) *Let $f \in \mathcal{E}'(\mathbb{R}^d)$. Then, there are a compact $K \Subset \mathbb{R}^d$ and $n \in \mathbb{N}$ such that $\text{supp } V_\psi f \subseteq K \times \mathbb{R}^d$ and*

$$\sup_{(x,\xi) \in K \times \mathbb{R}^d} \frac{|V_\psi f(x, \xi)|}{(1 + |\xi|)^n} < \infty.$$

In particular, $V_\psi f$ defines an element of $(\mathcal{E}(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}(\mathbb{R}_\xi^d))'$ via

$$\langle V_\psi f, \varphi \rangle := \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) \varphi(x, \xi) dx d\xi, \quad \varphi \in \mathcal{E}(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}(\mathbb{R}_\xi^d).$$

Proof. (i) It suffices to observe that we can factor $V_\psi = \mathcal{F}_{2\pi, t} \circ S$ through the continuous linear mappings

$$S : \mathcal{E}(\mathbb{R}_t^d) \rightarrow \mathcal{E}(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}(\mathbb{R}_t^d) : \varphi(t) \rightarrow \varphi(t) T_x \overline{\psi}(t)$$

and

$$\begin{aligned} \mathcal{F}_{2\pi, t} : \mathcal{E}(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}(\mathbb{R}_t^d) &\rightarrow \mathcal{E}(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}(\mathbb{R}_\xi^d) : \\ \varphi(x, t) &\rightarrow \int_{\mathbb{R}^d} \varphi(x, t) e^{-2\pi i t \xi} dt. \end{aligned}$$

(ii) Since $\text{supp } \overline{M_\xi T_x \psi} \subseteq \text{supp } \psi + x$ for all $(x, \xi) \in \mathbb{R}^{2d}$, we obtain that $V_\psi f(x, \xi) = \langle f, \overline{M_\xi T_x \psi} \rangle = 0$ for all $(x, \xi) \notin (\text{supp } f - \text{supp } \psi) \times \mathbb{R}^d$, that is, $\text{supp } V_\psi f \subseteq (\text{supp } f - \text{supp } \psi) \times \mathbb{R}^d$. The second part follows from Lemma 9.3.2. \square

Corollary 9.3.6. *Let $\psi \in \mathcal{D}(\mathbb{R}^d) \setminus \{0\}$ and let $\gamma \in \mathcal{D}(\mathbb{R}^d)$ be a synthesis window for ψ . Then, the desingularization formula (9.3.5) holds for all $f \in \mathcal{E}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{E}(\mathbb{R}^d)$.*

Proof. Choose a sequence $(\varphi_n)_n \subset \mathcal{D}(\mathbb{R}^d)$ such that $\varphi_n \rightarrow \varphi$ in $\mathcal{E}(\mathbb{R}^d)$. Hence, the desingularization formula (9.3.5) and Proposition 9.3.5 imply that

$$\langle f, \varphi \rangle = \lim_{n \rightarrow \infty} \langle f, \varphi_n \rangle = \lim_{n \rightarrow \infty} \langle V_\psi f, \overline{V_\gamma \varphi_n} \rangle = \langle V_\psi f, \overline{V_\gamma \varphi} \rangle.$$

□

9.3.2 The STFT on $\mathcal{S}'^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ and $\mathcal{S}'^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d)$

Let M_p and A_p be two weight sequences satisfying (9.2). We start with a discussion about the mapping properties of the STFT on the space $\mathcal{S}^*_{\dagger}(\mathbb{R}^d)$. Some preparation is needed. Let N_p and B_p be two other weight sequences satisfying (9.2). We denote by B the associated function of B_p . Let $d_1, d_2 \in \mathbb{N}$. For $h > 0$ we write X_h for the Banach space consisting of all $\varphi \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that

$$\sup_{(\alpha, \beta) \in \mathbb{N}^{d_1+d_2}} \sup_{(x, \xi) \in \mathbb{R}^{d_1+d_2}} \frac{|\partial_\xi^\beta \partial_x^\alpha \varphi(x, \xi)| e^{A(x/h) + B(\xi/h)}}{h^{|\alpha| + |\beta|} M_\alpha N_\beta} < \infty.$$

We have the following isomorphisms of l.c.s. (cf. [134, Prop. 2.12])

$$\mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^{d_1}) \widehat{\otimes} \mathcal{S}^{(N_p)}_{(B_p)}(\mathbb{R}^{d_2}) \cong \varprojlim_{h \rightarrow 0^+} X_h$$

and

$$\mathcal{S}^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^{d_1}) \widehat{\otimes} \mathcal{S}^{\{N_p\}}_{\{B_p\}}(\mathbb{R}^{d_2}) \cong \varinjlim_{h \rightarrow \infty} X_h.$$

In particular, $\mathcal{S}^*_{\dagger}(\mathbb{R}^{d_1}) \widehat{\otimes} \mathcal{S}^*_{\dagger}(\mathbb{R}^{d_2}) \cong \mathcal{S}^*_{\dagger}(\mathbb{R}^{d_1+d_2})$.

Proposition 9.3.7. *Let $\psi \in \mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$. Then,*

$$V_\psi : \mathcal{S}^*_{\dagger}(\mathbb{R}^d) \rightarrow \mathcal{S}^*_{\dagger}(\mathbb{R}^d_x) \widehat{\otimes} \mathcal{S}^{\dagger}_*(\mathbb{R}^d_\xi)$$

and

$$V_\psi^* : \mathcal{S}^*_{\dagger}(\mathbb{R}^d_x) \widehat{\otimes} \mathcal{S}^{\dagger}_*(\mathbb{R}^d_\xi) \rightarrow \mathcal{S}^*_{\dagger}(\mathbb{R}^d)$$

are well-defined continuous mappings.

Proof. We start by showing that V_ψ is well-defined and continuous. Consider the continuous linear mappings

$$S : \mathcal{S}_\dagger^*(\mathbb{R}_t^d) \rightarrow \mathcal{S}_\dagger^*(\mathbb{R}_{x,t}^{2d}) : \varphi(t) \rightarrow \varphi(t)T_x\overline{\psi}(t)$$

and

$$\mathcal{F}_{2\pi,t} : \mathcal{S}_\dagger^*(\mathbb{R}_{x,t}^{2d}) \rightarrow \mathcal{S}_\dagger^*(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}_*^\dagger(\mathbb{R}_\xi^d) : \varphi(x, t) \rightarrow \int_{\mathbb{R}^d} \varphi(x, t) e^{-2\pi i t \xi} d\xi.$$

The result now follows from the representation $V_\psi = \mathcal{F}_{2\pi,t} \circ S$. Next, we show that V_ψ^* is well-defined and continuous. Consider the continuous linear mappings

$$S : \mathcal{S}_\dagger^*(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}_*^\dagger(\mathbb{R}_\xi^d) \rightarrow \mathcal{S}_\dagger^*(\mathbb{R}_t^d) \widehat{\otimes} \mathcal{S}_\dagger^*(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}_*^\dagger(\mathbb{R}_\xi^d) : \\ \varphi(x, \xi) \rightarrow \varphi(x, \xi) M_\xi T_x \psi(t)$$

and

$$\text{id}_{\mathcal{S}_\dagger^*(\mathbb{R}_t^d)} \widehat{\otimes} 1(x) \widehat{\otimes} 1(\xi) : \mathcal{S}_\dagger^*(\mathbb{R}_t^d) \widehat{\otimes} \mathcal{S}_\dagger^*(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}_*^\dagger(\mathbb{R}_\xi^d) \rightarrow \mathcal{S}_\dagger^*(\mathbb{R}_t^d) : \\ \varphi(t, x, \xi) \rightarrow \int \int_{\mathbb{R}^{2d}} \varphi(t, x, \xi) dx d\xi.$$

The result now follows from the representation

$$V_\psi^* = \text{id}_{\mathcal{S}_\dagger^*(\mathbb{R}_t^d)} \widehat{\otimes} 1(x) \widehat{\otimes} 1(\xi) \circ S.$$

□

We are ready to define the STFT on the space $\mathcal{S}_\dagger'^*(\mathbb{R}^d)$. For $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $f \in \mathcal{S}_\dagger'^*(\mathbb{R}^d)$ we define

$$V_\psi f(x, \xi) := \langle f, \overline{M_\xi T_x \psi} \rangle = e^{-2\pi i x \xi} (f * \check{M_\xi \psi})(x), \quad (x, \xi) \in \mathbb{R}^{2d}.$$

Clearly, $V_\psi f$ is a continuous function on \mathbb{R}^{2d} .

Lemma 9.3.8. *Let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $f \in \mathcal{S}_\dagger'^*(\mathbb{R}^d)$. Then,*

$$\sup_{(x,\xi) \in \mathbb{R}^{2d}} \frac{|V_\psi f(x, \xi)|}{e^{A(x/h) + M(\xi/h)}} < \infty.$$

for some $h > 0$ (for all $h > 0$). In particular, $V_\psi f$ defines an element of $(\mathcal{S}_\dagger^*(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}_*^\dagger(\mathbb{R}_\xi^d))'$ via

$$\langle V_\psi f, \varphi \rangle := \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) \varphi(x, \xi) dx d\xi, \quad \varphi \in \mathcal{S}_\dagger^*(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}_*^\dagger(\mathbb{R}_\xi^d).$$

Proof. Let $h > 0$ be arbitrary. We have that

$$\begin{aligned} & \|M_\xi T_x \psi\|_{\mathcal{S}_{A_p, h}^{M_p, h}} \\ & \leq \sup_{\alpha \in \mathbb{N}^d} \sup_{t \in \mathbb{R}^d} \frac{e^{A(t/h)}}{h^{|\alpha|} M_\alpha} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (2\pi|\xi|)^{|\beta|} |\partial^{\alpha-\beta} \psi(t-x)| \\ & \leq \sup_{\alpha \in \mathbb{N}^d} \sup_{t \in \mathbb{R}^d} \frac{e^{A(2x/h)}}{2^{|\alpha|}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{(4\pi|\xi|/h)^{|\beta|}}{M_\beta} \frac{|\partial^{\alpha-\beta} \psi(t-x)| e^{A(2(t-x)/h)}}{(h/2)^{|\alpha|-|\beta|} M_{\alpha-\beta}} \\ & \leq \|\psi\|_{\mathcal{S}_{A_p, h/2}^{M_p, h/2}} e^{A(2x/h) + M(4\pi\xi/h)}. \end{aligned}$$

The result now follows from the continuity of f . \square

We define the adjoint STFT of $F \in (\mathcal{S}_\dagger^*(\mathbb{R}^d) \widehat{\otimes} \mathcal{S}_*^\dagger(\mathbb{R}^d))'$ as

$$\langle V_\psi^* F, \varphi \rangle := \langle F, \overline{V_\psi \varphi} \rangle, \quad \varphi \in \mathcal{S}_\dagger^*(\mathbb{R}^d).$$

Notice that $V_\psi^* F \in \mathcal{S}_\dagger'^*(\mathbb{R}^d)$ because of Proposition 9.3.7. By using similar arguments as in Subsection 9.3.1 one can now show that:

Proposition 9.3.9. *Let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. Then,*

$$V_\psi : \mathcal{S}_\dagger'^*(\mathbb{R}^d) \rightarrow (\mathcal{S}_\dagger^*(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}_*^\dagger(\mathbb{R}_\xi^d))'$$

and

$$V_\psi^* : (\mathcal{S}_\dagger^*(\mathbb{R}_x^d) \widehat{\otimes} \mathcal{S}_*^\dagger(\mathbb{R}_\xi^d))' \rightarrow \mathcal{S}_\dagger'^*(\mathbb{R}^d)$$

are well-defined continuous mappings. Moreover, if $\psi \neq 0$ and $\gamma \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ is a synthesis window for ψ , then

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \text{id}_{\mathcal{S}_\dagger'^*(\mathbb{R}^d)} \quad (9.3.6)$$

and the desingularization formula

$$\langle f, \varphi \rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\overline{\gamma}} \varphi(x, -\xi) dx d\xi \quad (9.3.7)$$

holds for all $f \in \mathcal{S}_\dagger'^*(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}_\dagger^*(\mathbb{R}^d)$.

Chapter 10

The smooth case

10.1 Introduction

The aim of this chapter is to present a detailed study of the structural and topological properties of a class of weighted L^1 convolutor spaces. More precisely, let $\mathcal{W} = (w_N)_N$ be an increasing weight system on \mathbb{R}^d and define $L^1_{\mathcal{W}}(\mathbb{R}^d)$ as the Fréchet space consisting of all measurable functions f on \mathbb{R}^d such that $\|fw_N\|_{L^1} < \infty$ for all $N \in \mathbb{N}$. We shall be concerned with the ensuing convolutor spaces

$$\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}) := \{f \in \mathcal{D}'(\mathbb{R}^d) \mid f * \varphi \in L^1_{\mathcal{W}}(\mathbb{R}^d) \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^d)\}$$

endowed with the initial topology with respect to the mapping

$$\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}) \rightarrow L_b(\mathcal{D}(\mathbb{R}^d), L^1_{\mathcal{W}}(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi)^1.$$

In order to be able to explain our results, we need to introduce some additional function spaces. Let v be a positive continuous function on \mathbb{R}^d . We write $\mathcal{B}_v(\mathbb{R}^d)$ for the Fréchet space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\max_{|\alpha| \leq n} \|\partial^\alpha \varphi\|_{Cv} < \infty$ for all $n \in \mathbb{N}$ and

¹Similarly as before, the continuity of the mapping $\varphi \rightarrow f * \varphi$ follows from the closed graph theorem.

$\dot{\mathcal{B}}_v(\mathbb{R}^d)$ for the closure of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{B}_v(\mathbb{R}^d)$. Next, let $\mathcal{V} = (v_N)_N$ be a decreasing weight system on \mathbb{R}^d . We introduce the (LF) -spaces

$$\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d) := \varinjlim_{N \in \mathbb{N}} \mathcal{B}_{v_N}(\mathbb{R}^d), \quad \dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d) := \varinjlim_{N \in \mathbb{N}} \dot{\mathcal{B}}_{v_N}(\mathbb{R}^d).$$

The main results of the present chapter can then be summarized as follows (recall that $\mathcal{W}^\circ = (1/w_N)_N$):

Theorem 10.1.1. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system on \mathbb{R}^d satisfying*

$$\forall N \exists M \geq N : \sup_{x \in \mathbb{R}^d} \frac{w_N(x + \cdot)}{w_M(x)} \in L_{\text{loc}}^\infty(\mathbb{R}^d).$$

Then, $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))' = \mathcal{O}'_C(\mathcal{D}, L_{\mathcal{W}}^1)$ algebraically. Moreover, the following statements are equivalent:

- (i) \mathcal{W}° satisfies (Ω) .
- (ii) $\mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d)$ is complete.
- (iii) $\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d)$ is complete.
- (iv) $\mathcal{O}'_C(\mathcal{D}, L_{\mathcal{W}}^1)$ is ultrabornological.
- (v) $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_b = \mathcal{O}'_C(\mathcal{D}, L_{\mathcal{W}}^1)$.

In such a case, the bidual of $\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d)$ is topologically isomorphic to $\mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d)$.

By applying Theorem 10.1.1 to a constant sequence $\mathcal{W} = (\omega)_N$, we obtain the following corollary.

Corollary 10.1.2. *Let ω be a positive measurable² function on \mathbb{R}^d satisfying*

$$\text{ess sup}_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L_{\text{loc}}^\infty(\mathbb{R}^d).$$

Set $(\dot{\mathcal{B}}_{1/\omega}(\mathbb{R}^d))'_b := \mathcal{D}'_{L_\omega^1}(\mathbb{R}^d)$. Then, the following properties hold:

²The fact that it suffices to assume that ω is merely measurable shall be shown in Theorem 10.4.1.

- (i) $\mathcal{D}'_{L^1_\omega}(\mathbb{R}^d) = \mathcal{O}'_C(\mathcal{D}, L^1_\omega)$ as locally convex spaces.
- (ii) The strong dual of $\mathcal{D}'_{L^1_\omega}(\mathbb{R}^d)$ is topologically isomorphic to $\mathcal{B}_{1/\omega}(\mathbb{R}^d)$.
- (iii) $\dot{\mathcal{B}}_{1/\omega}(\mathbb{R}^d)$ is a distinguished Fréchet space.

Theorem 10.4.1 with $\omega = 1$ is essentially due to Schwartz [151, pp. 201-203]. However, to the best of our knowledge, the topological identity $\mathcal{D}'_{L^1}(\mathbb{R}^d) = \mathcal{O}'_C(\mathcal{D}, L^1)$ remained unnoticed in the literature; Schwartz only showed that these spaces coincide algebraically and have the same bounded sets and null sequences [151, p. 202]. Structural and topological properties of weighted \mathcal{D}'_{L^1} spaces have recently been studied in the broader context of distribution spaces associated to general translation-invariant Banach spaces [51]. In particular, the analogues of Schwartz's results were proved there. As before, Corollary 10.1.2(i) seems to be new in this setting. We shall also show that, under natural assumptions on the increasing weight system $\mathcal{W} = (w_N)_N$, the space $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ coincides topologically with the space of convolutors of the Gelfand-Shilov space $\dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d) := \varprojlim_N \dot{\mathcal{B}}_{w_N}(\mathbb{R}^d)$. Hence, Theorem 10.1.1 comprises as well a quantified version of Grothendieck's results concerning $\mathcal{O}'_C(\mathbb{R}^d)$ (see Theorem 10.4.7).

This chapter is organized as follows. The (LF) -spaces $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$ and $\dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d)$ are discussed in Section 10.2. In particular, we shall show that these spaces are complete if and only if \mathcal{V} satisfies (Ω) . Interestingly, when specialized to the space $\mathcal{O}_C(\mathbb{R}^d)$, our result supplies what seems to be the first direct proof of the completeness of $\mathcal{O}_C(\mathbb{R}^d)$. For later use, we also characterize these spaces in terms of the STFT. In Section 10.3, we study various structural and topological properties of the space $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ via the STFT and complete the proof of Theorem 10.1.2. Finally, in Section 10.4.2, we present our results concerning weighted \mathcal{D}'_{L^1} spaces and apply our results to discuss the convolutor spaces of $\dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d)$.

10.2 Weighted inductive limits of spaces of smooth functions

In this section, we study the spaces $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$ and $\dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d)$. Our main goal is to show that all the regularity conditions from Subsection 2.3.1 are equivalent for these (LF) -spaces and that they are characterized by the condition (Ω) for \mathcal{V} . Furthermore, we also establish the mapping properties of the STFT on them.

Let v be a non-negative function on \mathbb{R}^d and let $n \in \mathbb{N}$. We define $\mathcal{B}_v^n(\mathbb{R}^d)$ as the seminormed space consisting of all $\varphi \in C^n(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{B}_v^n} := \max_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} |\partial^\alpha \varphi(x)| v(x) < \infty.$$

The closure of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{B}_v^n(\mathbb{R}^d)$ is denoted by $\dot{\mathcal{B}}_v^n(\mathbb{R}^d)$. Clearly, the latter space consists of all $\varphi \in C^n(\mathbb{R}^d)$ such that

$$\lim_{|x| \rightarrow \infty} |\partial^\alpha \varphi(x)| v(x) = 0$$

for all $|\alpha| \leq n$. If v is positive, then $\|\cdot\|_{\mathcal{B}_v^n}$ is actually a norm and if, in addition, $1/v$ is locally bounded, then $\mathcal{B}_v^n(\mathbb{R}^d)$ and $\dot{\mathcal{B}}_v^n(\mathbb{R}^d)$ are complete and, thus, Banach spaces. We set

$$\mathcal{B}_v(\mathbb{R}^d) := \varprojlim_{n \in \mathbb{N}} \mathcal{B}_v^n(\mathbb{R}^d), \quad \dot{\mathcal{B}}_v(\mathbb{R}^d) := \varprojlim_{n \in \mathbb{N}} \dot{\mathcal{B}}_v^n(\mathbb{R}^d).$$

Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system. We define the following (LF) -spaces

$$\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d) := \varinjlim_{N \in \mathbb{N}} \mathcal{B}_{v_N}(\mathbb{R}^d), \quad \dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d) := \varinjlim_{N \in \mathbb{N}} \dot{\mathcal{B}}_{v_N}(\mathbb{R}^d).$$

We shall often need to impose the following mild condition on \mathcal{V} :

$$\forall N \exists \tilde{N} \geq N : g_{N, \tilde{N}} := \sup_{x \in \mathbb{R}^d} \frac{v_{\tilde{N}}(x + \cdot)}{v_N(x)} \in L_{\text{loc}}^\infty(\mathbb{R}^d). \quad (10.2.1)$$

10.2.1 Regularity properties

The goal of this subsection is to show the following result:

Theorem 10.2.1. *Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system satisfying (10.2.1). Then, the following statements are equivalent:*

- (i) \mathcal{V} satisfies (Ω) .
- (ii) $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$ is boundedly retractive.
- (ii)' $\dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d)$ is boundedly retractive.
- (iii) $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$ satisfies (wQ) .
- (iii)' $\dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d)$ satisfies (wQ) .

The proof of Theorem 10.2.1 is based on Theorem 2.3.1, Lemma 9.1.13, and Gorny's inequality (cf. Lemma 3.4.6). We need two lemmas in preparation.

Lemma 10.2.2. *Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system satisfying (Ω) . Then, $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$ and $\dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d)$ are boundedly stable.*

Proof. Notice that $\dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d)$ is boundedly stable if $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$ is so. Hence, it suffices to show that $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$ is boundedly stable. Let $N \in \mathbb{N}$ be arbitrary and choose $M \geq N$ according to (Ω) . We shall show that, for all $K \geq M$, the spaces $\mathcal{B}_{v_K}(\mathbb{R}^d)$ and $\mathcal{B}_{v_M}(\mathbb{R}^d)$ induce the same topology on the bounded sets B of $\mathcal{B}_{v_N}(\mathbb{R}^d)$. We only need to prove that the topology induced by $\mathcal{B}_{v_K}(\mathbb{R}^d)$ is finer than the one induced by $\mathcal{B}_{v_M}(\mathbb{R}^d)$. Consider the basis of neighbourhoods of 0 in $\mathcal{B}_{v_M}(\mathbb{R}^d)$ given by

$$U(n, \varepsilon) = \{\varphi \in \mathcal{B}_{v_M}(\mathbb{R}^d) \mid \|\varphi\|_{\mathcal{B}_{v_M}^n} \leq \varepsilon\}, \quad n \in \mathbb{N}, \varepsilon > 0.$$

Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. Pick $\theta \in (0, 1)$ and $C > 0$ such that $v_M(x) \leq C v_N(x)^{1-\theta} v_K(x)^\theta$ for all $x \in \mathbb{R}^d$. Set

$$\delta = (\varepsilon/C)^{1/\theta} (\sup_{\varphi \in B} \|\varphi\|_{\mathcal{B}_{v_N}^n})^{-(1-\theta)/\theta}$$

and $V = \{\varphi \in \mathcal{B}_{v_K}(\mathbb{R}^d) \mid \|\varphi\|_{\mathcal{B}_{v_K}^n} \leq \delta\}$. We claim that $V \cap B \subseteq U(n, \varepsilon)$. Indeed, for $\varphi \in V \cap B$ it holds that

$$\begin{aligned} \|\varphi\|_{\mathcal{B}_{v_M}^n} &= \max_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} |\partial^\alpha \varphi(x)| v_M(x) \\ &\leq C \max_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} (|\partial^\alpha \varphi(x)| v_N(x))^{1-\theta} (|\partial^\alpha \varphi(x)| v_K(x))^\theta \\ &\leq C \|\varphi\|_{\mathcal{B}_{v_N}^n}^{1-\theta} \|\varphi\|_{\mathcal{B}_{v_K}^n}^\theta \\ &\leq \varepsilon. \end{aligned}$$

□

Lemma 10.2.3. *Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system satisfying (10.2.1) and (Ω) . Then,*

$$\begin{aligned} \forall N \exists M \geq N \forall K \geq M \forall m \exists k \exists \theta \in (0, 1) \exists C > 0 \forall \varphi \in \mathcal{B}_{v_N}(\mathbb{R}^d) : \\ \|\varphi\|_{\mathcal{B}_{v_M}^m} \leq C \|\varphi\|_{C v_N}^{1-\theta} \|\varphi\|_{\mathcal{B}_{v_K}^k}^\theta. \end{aligned}$$

Proof. The proof is based on the following consequence of Lemma 3.4.6: For all $k, m \in \mathbb{N}$ with $0 < m \leq k$ there is $C > 0$ such that

$$\|\varphi\|_{C^m([-1,1]^d)} \leq C \|\varphi\|_{C([-2,2]^d)}^{1-m/k} \|\varphi\|_{C^k([-2,2]^d)}^{m/k}$$

for all $\varphi \in C^k([-2,2]^d)$. Let $N \in \mathbb{N}$ be arbitrary and choose $\tilde{N} \geq N$ as in (10.2.1). Next, select $M \geq \tilde{N}$ according to (Ω) . Let $K \geq M$ and $m \in \mathbb{N}$ be arbitrary. Choose $\tilde{K} \geq K$ as in (10.2.1). Let $\theta \in (0, 1)$ and $C' > 0$ be such that $v_M(x) \leq C' v_{\tilde{N}}(x)^{1-\theta} v_{\tilde{K}}(x)^\theta$ for all $x \in \mathbb{R}^d$. Finally, pick $k \geq m$ so large that $m/k \leq \theta$. Notice that for $0 \leq a \leq b$ it holds that $a^{1-m/k} b^{m/k} \leq a^{1-\theta} b^\theta$. Hence,

$$\begin{aligned} \|\varphi\|_{\mathcal{B}_{v_M}^m} &\leq \sup_{x \in \mathbb{R}^d} \|T_{-x} \varphi\|_{C^m([-1,1]^d)} v_M(x) \\ &\leq C C' \sup_{x \in \mathbb{R}^d} \|T_{-x} \varphi\|_{C([-2,2]^d)}^{1-m/k} \|T_{-x} \varphi\|_{C^k([-2,2]^d)}^{m/k} v_{\tilde{N}}(x)^{1-\theta} v_{\tilde{K}}(x)^\theta \\ &\leq C C' \left(\sup_{x \in \mathbb{R}^d} \|T_{-x} \varphi\|_{C([-2,2]^d)} v_{\tilde{N}}(x) \right)^{1-\theta} \\ &\quad \left(\sup_{x \in \mathbb{R}^d} \|T_{-x} \varphi\|_{C^k([-2,2]^d)} v_{\tilde{K}}(x) \right)^\theta \end{aligned}$$

for all $\varphi \in \mathcal{B}_{v_N}(\mathbb{R}^d)$. The result now follows from the fact that

$$\sup_{x \in \mathbb{R}^d} \|T_{-x}\varphi\|_{C^l([-2,2]^d)} v_{\tilde{L}}(x) \leq \|\check{g}_{L,\tilde{L}}\|_{L^\infty([-2,2]^d)} \|\varphi\|_{\mathcal{B}_{v_L}^l}$$

for all $l, L \in \mathbb{N}$ and $\varphi \in \mathcal{B}_{v_L}(\mathbb{R}^d)$. \square

Proof of Theorem 10.2.1. The implications $(ii) \Rightarrow (iii)$ and $(ii)' \Rightarrow (iii)'$ hold for general (LF) -spaces (cf. Subsection 2.3.1). Observe that $\dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d)$ satisfies (wQ) if $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$ does so, that is, $(iii) \Rightarrow (iii)'$. Next, in view of Theorem 2.3.1, $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (ii)'$ follow directly from Lemmas 10.2.2 and 10.2.3. Finally, we show $(iii)' \Rightarrow (i)$. Since $\mathcal{D}_{[-1,1]^d} \cong s$ [114, Prop. 31.12], Remark 9.1.10 and Lemma 9.1.13 imply that it suffices to show that $(\mathcal{D}_{[-1,1]^d}, \mathcal{V})$ satisfies $(S_2)^*$. Let $N \in \mathbb{N}$ be arbitrary and choose $\tilde{N} \geq N$ as in (10.2.1). Next, choose $M \geq \tilde{N}$ and $n \in \mathbb{N}$ according to (wQ) . Pick $\tilde{M} \geq M$ as in (10.2.1). We shall show $(S_2)^*$ for \tilde{M} and n . Let $K \geq \tilde{M}$ and $m \in \mathbb{N}$ be arbitrary. Choose $\tilde{K} \geq K$ as in (10.2.1). By (wQ) there are $k \in \mathbb{N}$ and $C > 0$ such that

$$\|\varphi\|_{\mathcal{B}_{v_M}^m} \leq C \left(\|\varphi\|_{\mathcal{B}_{v_{\tilde{N}}}^n} + \|\varphi\|_{\mathcal{B}_{v_{\tilde{K}}}^k} \right)$$

for all $\varphi \in \dot{\mathcal{B}}_{v_N}(\mathbb{R}^d)$. Let $x \in \mathbb{R}^d$ and $\chi \in \mathcal{D}_{[-1,1]^d}$ be arbitrary. For all $l, L \in \mathbb{N}$ we have that $T_x\chi \in \mathcal{D}(\mathbb{R}^d) \subset \dot{\mathcal{B}}_{v_L}(\mathbb{R}^d)$,

$$\|T_x\chi\|_{\mathcal{B}_{v_{\tilde{L}}}^l} \leq \|g_{L,\tilde{L}}\|_{L^\infty([-1,1]^d)} v_L(x) \|\chi\|_{C^l([-1,1]^d)}$$

and

$$v_{\tilde{L}}(x) \|\chi\|_{C^l([-1,1]^d)} \leq \|\check{g}_{L,\tilde{L}}\|_{L^\infty([-1,1]^d)} \|T_x\chi\|_{\mathcal{B}_{v_L}^l}.$$

Hence,

$$\begin{aligned} v_{\tilde{M}}(x) \|\chi\|_{C^m([-1,1]^d)} &\leq \|\check{g}_{M,\tilde{M}}\|_{L^\infty([-1,1]^d)} \|T_x\chi\|_{\mathcal{B}_{v_M}^m} \\ &\leq C \|\check{g}_{M,\tilde{M}}\|_{L^\infty([-1,1]^d)} (\|T_x\chi\|_{\mathcal{B}_{v_{\tilde{N}}}^n} + \|T_x\chi\|_{\mathcal{B}_{v_{\tilde{K}}}^k}) \\ &\leq C' (v_N(x) \|\chi\|_{C^n([-1,1]^d)} + v_K(x) \|\chi\|_{C^k([-1,1]^d)}), \end{aligned}$$

where the constant C' is given by

$$\|\check{g}_{M,\tilde{M}}\|_{L^\infty([-1,1]^d)} \max\{\|g_{N,\tilde{N}}\|_{L^\infty([-1,1]^d)}, \|g_{K,\tilde{K}}\|_{L^\infty([-1,1]^d)}\} < \infty.$$

\square

10.2.2 Characterization via the STFT

We now turn our attention to the mapping properties of the STFT on the spaces $\mathcal{B}_V(\mathbb{R}^d)$ and $\dot{\mathcal{B}}_V(\mathbb{R}^d)$. The following two technical lemmas are needed.

Lemma 10.2.4. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and let v and w be positive measurable functions on \mathbb{R}^d such that*

$$g := \sup_{x \in \mathbb{R}^d} \frac{v(x + \cdot)}{w(x)} \in L_{\text{loc}}^\infty(\mathbb{R}^d). \quad (10.2.2)$$

Then,

$$V_\psi : \mathcal{B}_w^n(\mathbb{R}^d) \rightarrow C v \otimes (1 + |\cdot|)^n(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi : \dot{\mathcal{B}}_w^n(\mathbb{R}^d) \rightarrow C(v \otimes (1 + |\cdot|)^n)_0(\mathbb{R}_{x,\xi}^{2d})$$

are well-defined continuous mappings.

Proof. Set $K = \text{supp } \psi$ and let $\varphi \in \mathcal{B}_w^n(\mathbb{R}^d)$ be arbitrary. Then,

$$\begin{aligned} & |V_\psi \varphi(x, \xi)| v(x) (1 + |\xi|)^n \\ & \leq (1 + \sqrt{d})^n \max_{|\alpha| \leq n} |\xi^\alpha V_\psi \varphi(x, \xi)| v(x) \\ & \leq (1 + \sqrt{d})^n \max_{|\alpha| \leq n} (2\pi)^{-|\alpha|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_K |\partial^\beta \varphi(t + x)| \cdot \\ & \quad w(t + x) \check{g}(t) |\partial^{\alpha-\beta} \psi(t)| dt \\ & \leq (1 + \sqrt{d})^n |K| \|\psi\|_{C^n(K)} \|\check{g}\|_{L^\infty(K)} \max_{|\alpha| \leq n} \sup_{t \in K} |\partial^\alpha \varphi(t + x)| w(t + x) \end{aligned}$$

for all $(x, \xi) \in \mathbb{R}^{2d}$, which shows the continuity of V_ψ on $\mathcal{B}_w^n(\mathbb{R}^d)$. We still need to show that $V_\psi(\dot{\mathcal{B}}_w^n(\mathbb{R}^d)) \subseteq C(v \otimes (1 + |\cdot|)^n)_0(\mathbb{R}_{x,\xi}^{2d})$. Let $\varphi \in \dot{\mathcal{B}}_w^n(\mathbb{R}^d)$ be arbitrary. The above inequality shows that

$$\lim_{|x| \rightarrow \infty} \sup_{\xi \in \mathbb{R}^d} |V_\psi \varphi(x, \xi)| v(x) (1 + |\xi|)^n = 0.$$

Therefore, we only need to prove that

$$\lim_{|\xi| \rightarrow \infty} \sup_{x \in K'} |V_\psi \varphi(x, \xi)| v(x) (1 + |\xi|)^n = 0$$

for all compact subsets $K' \Subset \mathbb{R}^d$. Since

$$\begin{aligned} \sup_{x \in K'} |V_\psi \varphi(x, \xi)| v(x) (1 + |\xi|)^n &\leq (1 + \sqrt{d})^n \|v\|_{L^\infty(K')}. \\ \sup_{x \in K'} \max_{|\alpha| \leq n} (2\pi)^{-|\alpha|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\mathcal{F}(\partial^\beta \varphi T_x \partial^{\alpha-\beta} \bar{\psi})(2\pi\xi)|, \end{aligned}$$

it suffices to show that

$$\begin{aligned} &\lim_{|\xi| \rightarrow \infty} \sup_{x \in K'} |\mathcal{F}(f T_x \chi)(2\pi\xi)| \\ &= \lim_{|\xi| \rightarrow \infty} \sup_{x \in K'} |\langle e^{-2\pi i t \xi}, f(t) T_x \chi(t) \rangle_{(L^\infty, L^1)}| = 0 \end{aligned}$$

for all $f \in C(\mathbb{R}^d)$ and $\chi \in \mathcal{D}(\mathbb{R}^d)$. Since the set $\{e^{-2\pi i t \xi} \mid \xi \in \mathbb{R}^d\}$ is bounded in $L^\infty(\mathbb{R}_t^d)$ and $\lim_{|\xi| \rightarrow \infty} e^{-2\pi i t \xi} = 0$ in $L^\infty(\mathbb{R}_t^d)$ endowed with the weak-* topology (Riemann-Lebesgue lemma), we obtain that $\lim_{|\xi| \rightarrow \infty} e^{-2\pi i t \xi} = 0$ on compact subsets of $L^1(\mathbb{R}^d)$. The result now follows by observing that the set $\{f T_x \chi \mid x \in K'\}$ is compact in $L^1(\mathbb{R}^d)$. \square

Lemma 10.2.5. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and let v and w be positive measurable functions on \mathbb{R}^d satisfying (10.2.2). Then,*

$$V_\psi^* : Cw \otimes (1 + |\cdot|)^{n+d+1}(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \mathcal{B}_v^n(\mathbb{R}^d)$$

and

$$V_\psi^* : C(w \otimes (1 + |\cdot|)^{n+d+1})_0(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \dot{\mathcal{B}}_v^n(\mathbb{R}^d)$$

are well-defined continuous mappings.

Proof. Set $K = \text{supp } \psi$ and let $F \in Cw \otimes (1 + |\cdot|)^{n+d+1}(\mathbb{R}_{x,\xi}^{2d})$ be arbitrary. Then,

$$\begin{aligned} &|\partial^\alpha V_\psi^* F(t)| v(t) \\ &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} v(t) \int \int_{\mathbb{R}^{2d}} |F(x, \xi)| (2\pi|\xi|)^{|\beta|} |\partial^{\alpha-\beta} \psi(t-x)| dx d\xi \\ &\leq (2\pi)^n |K| \|\psi\|_{C^n(K)} \|g\|_{L^\infty(K)} \\ &\int_{\mathbb{R}^d} \sup_{x \in K} \frac{|F(t-x, \xi)| w(t-x) (1 + |\xi|)^{n+d+1}}{(1 + |\xi|)^{d+1}} d\xi \end{aligned}$$

for all $t \in \mathbb{R}^d$ and $|\alpha| \leq n$, which shows that V_ψ acts continuously on $Cw \otimes (1 + |\cdot|)^{n+d+1}(\mathbb{R}_{x,\xi}^{2d})$. The inclusion $V_\psi^*(C(w \otimes (1 + |\cdot|)^{n+d+1})_0(\mathbb{R}_{x,\xi}^{2d})) \subseteq \dot{\mathcal{B}}_v^n(\mathbb{R}^d)$ follows from the above inequality and Lebesgue's dominated convergence theorem. \square

We define $\mathcal{W}_{\text{pol}} := ((1 + |\cdot|)^n)_n$, an increasing weight system on \mathbb{R}^d . Lemmas 10.2.4 and 10.2.5 directly imply the following result.

Proposition 10.2.6. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and let $\mathcal{V} = (v_N)_N$ be a decreasing weight system satisfying (10.2.1). Then,*

$$V_\psi : \mathcal{B}_{\mathcal{V}}(\mathbb{R}^d) \rightarrow \mathcal{V} \otimes_i \mathcal{W}_{\text{pol}} C(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi^* : \mathcal{V} \otimes_i \mathcal{W}_{\text{pol}} C(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$$

are well-defined continuous mappings.

Proposition 10.2.7. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and let $\mathcal{V} = (v_N)_N$ be a decreasing weight system satisfying (10.2.1). Then,*

$$V_\psi : \dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d) \rightarrow (\mathcal{V} \otimes_i \mathcal{W}_{\text{pol}})_0 C(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi^* : (\mathcal{V} \otimes_i \mathcal{W}_{\text{pol}})_0 C(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \dot{\mathcal{B}}_{\mathcal{V}}(\mathbb{R}^d)$$

are well-defined continuous mappings.

Suppose that \mathcal{V} satisfies (10.2.1) and (Ω) . By combining Proposition 10.2.6 and (9.3.4) with Corollary 9.1.17, we obtain an explicit system of seminorms generating the topology of $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$; this will be of vital importance later on. More precisely, we have that:

Corollary 10.2.8. *Let $\psi \in \mathcal{D}(\mathbb{R}^d) \setminus \{0\}$ and let $\mathcal{V} = (v_N)_N$ be a decreasing weight system satisfying (10.2.1) and (Ω) . Then, $f \in \mathcal{D}'(\mathbb{R}^d)$ belongs to $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$ if and only if $\|V_\psi f\|_{C_u} < \infty$ for all $u \in \overline{U}(\mathcal{V} \otimes_i \mathcal{W}_{\text{pol}})$. Moreover, the topology of $\mathcal{B}_{\mathcal{V}}(\mathbb{R}^d)$ is generated by the system of seminorms $\{\|V_\psi(\cdot)\|_{C_u} \mid u \in \overline{U}(\mathcal{V} \otimes_i \mathcal{W}_{\text{pol}})\}$.*

10.3 On a class of weighted L^1 convolutor spaces

We are ready to study the convolutor spaces $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$. In the main theorem of this section, we show that $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ is ultrabornological if and only if \mathcal{W}° satisfies (Ω) . To achieve this goal, we first establish various structural and topological properties of this space. Since the latter will rely heavily on the mapping properties of the STFT on $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$, we start with a discussion about the STFT on this space.

Let w be a positive measurable function on \mathbb{R}^d . We define $L^1_w(\mathbb{R}^d)$ as the Banach space consisting of all measurable functions f on \mathbb{R}^d such that $\|f\|_{L^1_w} := \|fw\|_{L^1} < \infty$. For an increasing weight system $\mathcal{W} = (w_N)_N$ we set

$$L^1_{\mathcal{W}}(\mathbb{R}^d) := \varprojlim_{N \in \mathbb{N}} L^1_{w_N}(\mathbb{R}^d).$$

Furthermore, we define $\mathcal{D}_{L^1_{\mathcal{W}}}(\mathbb{R}^d)$ as the Fréchet space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\|\varphi\|_{\mathcal{D}_{L^1_{w_N}}} := \max_{|\alpha| \leq n} \|\partial^\alpha \varphi\|_{L^1_{w_N}} < \infty$ for all $n, N \in \mathbb{N}$. We shall be concerned with the following convolutor spaces

$$\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}) := \{f \in \mathcal{D}'(\mathbb{R}^d) \mid f * \varphi \in L^1_{\mathcal{W}}(\mathbb{R}^d) \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^d)\}$$

endowed with the initial topology with respect to the mapping

$$\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}) \rightarrow L_b(\mathcal{D}(\mathbb{R}^d), L^1_{\mathcal{W}}(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi).$$

Finally, throughout this section we shall often need to impose the following mild condition on \mathcal{W} (cf. (10.2.1)):

$$\forall N \exists \tilde{N} \geq N : h_{N, \tilde{N}} := \sup_{x \in \mathbb{R}^d} \frac{w_N(x + \cdot)}{w_{\tilde{N}}(x)} \in L^\infty_{\text{loc}}(\mathbb{R}^d). \quad (10.3.1)$$

10.3.1 Characterization via the STFT

In this subsection, the mapping properties of the STFT on the space $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ are established. Some preparation is needed. We define $\mathcal{V}_{\text{pol}} := ((1 + |\cdot|)^{-n})_n$, a decreasing weight system on \mathbb{R}^d . Let $\mathcal{W} = (w_N)_N$ be an increasing weight system. By [92, Prop. 1.5] and [13, Thm. 3.1(d)] we may identify $L^1_{\mathcal{W}}(\mathbb{R}^d_x) \widehat{\otimes}_{\varepsilon} \mathcal{V}_{\text{pol}} C(\mathbb{R}^d_{\xi})$ with the space consisting of all $f \in C(\mathbb{R}^d_{\xi}, L^1_{\mathcal{W}}(\mathbb{R}^d_x))$ satisfying the ensuing property: For all $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that

$$\sup_{\xi \in \mathbb{R}^d} \frac{\|f(\cdot, \xi)\|_{L^1_{w_N}}} {(1 + |\xi|)^n} < \infty.$$

Hence, $f \in C(\mathbb{R}^d_{\xi}, L^1_{\mathcal{W}}(\mathbb{R}^d_x))$ belongs to $L^1_{\mathcal{W}}(\mathbb{R}^d_x) \widehat{\otimes}_{\varepsilon} \mathcal{V}_{\text{pol}} C(\mathbb{R}^d_{\xi})$ if and only if

$$\|f\|_{L^1_{w_N}, v} := \sup_{\xi \in \mathbb{R}^d} \|f(\cdot, \xi)\|_{L^1_{w_N}} v(\xi) < \infty$$

for all $N \in \mathbb{N}$ and $v \in \overline{\mathcal{V}}_{\text{pol}}$. Moreover, by [13, Thm. 3.1(c)], the topology of $L^1_{\mathcal{W}}(\mathbb{R}^d_x) \widehat{\otimes}_{\varepsilon} \mathcal{V}_{\text{pol}} C(\mathbb{R}^d_{\xi})$ is generated by the system of seminorms $\{\|\cdot\|_{L^1_{w_N}, v} \mid N \in \mathbb{N}, v \in \overline{\mathcal{V}}_{\text{pol}}\}$. We then have:

Proposition 10.3.1. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1). Then,*

$$V_{\psi} : \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}) \rightarrow L^1_{\mathcal{W}}(\mathbb{R}^d_x) \widehat{\otimes}_{\varepsilon} \mathcal{V}_{\text{pol}} C(\mathbb{R}^d_{\xi})$$

and

$$V_{\psi}^* : L^1_{\mathcal{W}}(\mathbb{R}^d_x) \widehat{\otimes}_{\varepsilon} \mathcal{V}_{\text{pol}} C(\mathbb{R}^d_{\xi}) \rightarrow \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$$

are well-defined continuous mappings.

Proof. We first consider V_{ψ} . Let $f \in \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ be arbitrary. Observe that $V_{\psi} f(\cdot, \xi) \in L^1_{\mathcal{W}}(\mathbb{R}^d)$ for $\xi \in \mathbb{R}^d$ fixed, as follows from the representation $V_{\psi} f(x, \xi) = e^{-2\pi i x \xi} (f * M_{\xi} \check{\psi})(x)$. We now prove that $\mathbb{R}^d \rightarrow L^1_{\mathcal{W}}(\mathbb{R}^d) : \xi \rightarrow V_{\psi} f(\cdot, \xi)$ is continuous. Since the mappings $\mathbb{R}^d \rightarrow \mathcal{D}(\mathbb{R}^d) : \xi \rightarrow M_{\xi} \check{\psi}$ and $\mathcal{D}(\mathbb{R}^d) \rightarrow L^1_{\mathcal{W}}(\mathbb{R}^d) : \varphi \rightarrow f * \varphi$ are continuous, the mapping

$$\mathbb{R}^d \rightarrow L^1_{\mathcal{W}}(\mathbb{R}^d) : \xi \rightarrow f * M_{\xi} \check{\psi} \quad (10.3.2)$$

is also continuous. Let $\xi_0, \xi \in \mathbb{R}^d$ and $N \in \mathbb{N}$ be arbitrary. We have that

$$\begin{aligned}
& \|V_\psi f(x, \xi) - V_\psi f(x, \xi_0)\|_{L^1_{w_N}, x} \\
&= \|e^{-2\pi i x \xi} (f * M_\xi \check{\psi})(x) - e^{-2\pi i x \xi_0} (f * M_{\xi_0} \check{\psi})(x)\|_{L^1_{w_N}, x} \\
&\leq \|f * M_\xi \check{\psi} - f * M_{\xi_0} \check{\psi}\|_{L^1_{w_N}} \\
&+ \|(e^{-2\pi i x \xi} - e^{-2\pi i x \xi_0})(f * M_{\xi_0} \check{\psi})(x)\|_{L^1_{w_N}, x}.
\end{aligned}$$

The first term tends to zero as $\xi \rightarrow \xi_0$ because the mapping (10.3.2) is continuous while the second term tends to zero as $\xi \rightarrow \xi_0$ because of Lebesgue's dominated convergence theorem. Let $N \in \mathbb{N}$ and $v \in \overline{V}(\mathcal{V}_{\text{pol}})$ be arbitrary. The set $B = \{M_\xi \check{\psi} v(\xi) \mid \xi \in \mathbb{R}^d\}$ is bounded in $\mathcal{D}(\mathbb{R}^d)$. Hence,

$$\|V_\psi f\|_{L^1_{w_N}, v} = \sup_{\xi \in \mathbb{R}^d} \|f * M_\xi \check{\psi}\|_{L^1_{w_N}} v(\xi) = \sup_{\varphi \in B} \|f * \varphi\|_{L^1_{w_N}},$$

which shows that V_ψ is well-defined and continuous. Next, we treat V_ψ^* . Let $B \subset \mathcal{D}(\mathbb{R}^d)$ bounded and $N \in \mathbb{N}$ be arbitrary. Choose $\tilde{N} \geq N$ according to (10.3.1). Proposition 9.3.1 implies that there are $K \Subset \mathbb{R}^d$ and $v \in \overline{V}(\mathcal{V}_{\text{pol}})$ such that $\text{supp } V_\psi \check{\varphi} \subseteq K \times \mathbb{R}^d$ and $|V_\psi \check{\varphi}(x, \xi)| \leq v(\xi)$, $(x, \xi) \in K \times \mathbb{R}^d$, for all $\varphi \in B$. Set $\bar{v} = v(\cdot)(1 + |\cdot|)^{d+1} \in \overline{V}(\mathcal{V}_{\text{pol}})$. Hence,

$$\begin{aligned}
& \sup_{\varphi \in B} \|V_\psi^* F * \varphi\|_{L^1_{w_N}} \\
&\leq \sup_{\varphi \in B} \int \int \int_{\mathbb{R}^{3d}} |F(x, \xi)| |V_\psi \check{\varphi}(x - t, \xi)| w_N(t) dx d\xi dt \\
&\leq |K| \max_{x \in K} \int \int_{\mathbb{R}^{2d}} |F(x + t, \xi)| v(\xi) w_N(t) d\xi dt \\
&\leq C \|F\|_{\bar{v}, L^1_{w_{\tilde{N}}}}
\end{aligned}$$

for all $F \in L^1_{\mathcal{W}}(\mathbb{R}^d_x) \widehat{\otimes}_\varepsilon \mathcal{V}_{\text{pol}} C(\mathbb{R}^d_\xi)$, where

$$C = |K| \|\check{h}_{N, \tilde{N}}\|_{L^\infty(K)} \int_{\mathbb{R}^d} (1 + |\xi|)^{-(d+1)} d\xi < \infty.$$

This shows that V_ψ is well-defined and continuous. □

Proposition 10.3.1 and (9.3.4) imply the following corollary.

Corollary 10.3.2. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1). Then, $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ is complete.*

10.3.2 A predual

Our next goal is to show that the dual of $\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d)$, endowed with a suitable \mathfrak{S} -topology, is isomorphic to $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$. We start by introducing the following general notion: Let $E = \varinjlim_N E_N$ be an (LF) -space and set

$$\mathfrak{S} = \{B \subset E \mid B \text{ is contained and bounded in } E_N \\ \text{for some } N \in \mathbb{N}\}.$$

We write $bs(E', E)$ for the \mathfrak{S} -topology on E' , that is, the topology of uniform convergence on sets of \mathfrak{S} . Grothendieck's factorization theorem implies that $bs(E', E)$ does not depend on the defining inductive spectrum of E . Clearly, $bs(E', E) = b(E', E)$ if E is regular.

Theorem 10.3.3. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1). Then, $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_{bs} = \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$.*

We need some results in preparation for the proof of Theorem 10.3.3.

Proposition 10.3.4. *Let w be a positive continuous function on \mathbb{R}^d and let $f \in (\dot{\mathcal{B}}_{1/w}(\mathbb{R}^d))'$. Then, there are $n \in \mathbb{N}$ and regular complex Borel measures $\mu_\alpha \in (C_0(\mathbb{R}^d))'$, $|\alpha| \leq n$, such that*

$$\langle f, \varphi \rangle = \sum_{|\alpha| \leq n} (-1)^{|\alpha|} \int_{\mathbb{R}^d} \frac{\partial^\alpha \varphi(x)}{w(x)} d\mu_\alpha(x), \quad \varphi \in \dot{\mathcal{B}}_{1/w}(\mathbb{R}^d). \quad (10.3.3)$$

Proof. The distribution f can be extended to a continuous linear functional on $\dot{\mathcal{B}}_{1/w}^n(\mathbb{R}^d)$ for some $n \in \mathbb{N}$ (we also denote this extension by f). Consider the topological embedding

$$\iota : \dot{\mathcal{B}}_{1/w}^n(\mathbb{R}^d) \rightarrow \bigoplus_{|\alpha| \leq n} C_0(\mathbb{R}^d) : \varphi \rightarrow \left(\frac{\partial^\alpha \varphi}{w} \right)_\alpha$$

and let $\rho : \iota(\dot{\mathcal{B}}_{1/w}^n(\mathbb{R}^d)) \rightarrow \dot{\mathcal{B}}_{1/w}^n(\mathbb{R}^d)$ be the continuous linear mapping such that $\rho \circ \iota = \text{id}$. By the Hahn-Banach theorem the continuous linear functional $f \circ \rho$ can be extended to an element \tilde{f} of $(\bigoplus_{|\alpha| \leq n} C_0(\mathbb{R}^d))' \cong \bigoplus_{|\alpha| \leq n} (C_0(\mathbb{R}^d))'$. The Riesz representation theorem [143, Thm. 6.9] implies that there are regular complex Borel measures $\mu_\alpha \in (C_0(\mathbb{R}^d))'$, $|\alpha| \leq n$, such that

$$\langle \tilde{f}, (g_\alpha)_\alpha \rangle = \sum_{|\alpha| \leq n} (-1)^{|\alpha|} \int_{\mathbb{R}^d} g_\alpha(x) d\mu_\alpha(x), \quad (g_\alpha)_\alpha \in \bigoplus_{|\alpha| \leq n} C_0(\mathbb{R}^d).$$

Hence,

$$\begin{aligned} \langle f, \varphi \rangle &= \langle f \circ \rho, \iota(\varphi) \rangle = \langle \tilde{f}, (\partial^\alpha \varphi / w)_\alpha \rangle \\ &= \sum_{|\alpha| \leq n} (-1)^{|\alpha|} \int_{\mathbb{R}^d} \frac{\partial^\alpha \varphi(x)}{w(x)} d\mu_\alpha(x) \end{aligned}$$

for all $\varphi \in \dot{\mathcal{B}}_{1/w}(\mathbb{R}^d)$. □

Corollary 10.3.5. *Let v and w be positive continuous functions on \mathbb{R}^d satisfying (10.2.2) and let $f \in (\dot{\mathcal{B}}_{1/w}(\mathbb{R}^d))'$.*

(i) $f * \varphi \in L_v^1(\mathbb{R}^d)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

(ii) For all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and $h \in C(1/v)_0(\mathbb{R}^d)$ it holds that $\check{\varphi} * h \in \dot{\mathcal{B}}_{1/w}(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f * \varphi(x) h(x) dx = \langle f, \check{\varphi} * h \rangle.$$

Proof. Suppose that f can be represented as (10.3.3). In particular,

$$f * \varphi = \sum_{|\alpha| \leq n} \int_{\mathbb{R}^d} \frac{\partial^\alpha \varphi(\cdot - t)}{w(t)} d\mu_\alpha(t), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

(i) Set $K = \text{supp } \varphi$. We have that

$$\|f * \varphi\|_{L_v^1} \leq \sum_{|\alpha| \leq n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\partial^\alpha \varphi(x - t)|}{w(t)} d|\mu_\alpha|(t) v(x) dx,$$

where $|\mu_\alpha|$ denotes the total variation measure associated with μ_α . For each $|\alpha| \leq n$ it holds that

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial^\alpha \varphi(x - t)| v(x) dx \frac{1}{w(t)} d|\mu_\alpha|(t) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial^\alpha \varphi(x)| v(x + t) dx \frac{1}{w(t)} d|\mu_\alpha|(t) \\ &\leq |K| \|\varphi\|_{C^n(K)} \|g\|_{L^\infty(K)} |\mu_\alpha|(\mathbb{R}^d) < \infty. \end{aligned}$$

Hence, Fubini's theorem implies that $\|f * \varphi\|_{L_v^1} < \infty$.

(ii) Set $K = \text{supp } \varphi$. For all $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$ we have that

$$\frac{|\partial^\alpha (\check{\varphi} * h)(x)|}{w(x)} \leq \|\partial^\alpha \varphi\|_{C(K)} \|g\|_{L^\infty(K)} \sup_{t \in K} \frac{h(t + x)}{v(t + x)}, \quad (10.3.4)$$

which tends to zero as $|x| \rightarrow \infty$, that is, $\check{\varphi} * h \in \dot{B}_{1/w}(\mathbb{R}^d)$. Finally,

$$\begin{aligned} \int_{\mathbb{R}^d} f * \varphi(x) h(x) dx &= \sum_{|\alpha| \leq n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\partial^\alpha \varphi(x - t)}{w(t)} d\mu_\alpha(t) h(x) dx \\ &= \sum_{|\alpha| \leq n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial^\alpha \varphi(x - t) h(x) dx \frac{1}{w(t)} d\mu_\alpha(t) \\ &= \sum_{|\alpha| \leq n} (-1)^{|\alpha|} \int_{\mathbb{R}^d} \frac{\partial^\alpha (\check{\varphi} * h)(t)}{w(t)} d\mu_\alpha(t) \\ &= \langle f, \check{\varphi} * h \rangle. \end{aligned}$$

□

Proof of Theorem 10.3.3. From Corollary 10.3.5(i) we obtain that $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))' \subseteq \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$. We now show that this inclusion holds continuously if we endow the former space with the *bs*-topology. For this, we need the following result from measure theory (cf. [143, Thm. 6.9 and Thm. 6.13]): Let w be a positive continuous function on \mathbb{R}^d and let $f \in L^1_w(\mathbb{R}^d)$. Denote by B the unit ball in $C_0(\mathbb{R}^d)$. Then,

$$\|f\|_{L^1_w} = \sup_{h \in B} \left| \int_{\mathbb{R}^d} f(x)h(x)w(x)dx \right|.$$

Let $B' \subset \mathcal{D}(\mathbb{R}^d)$ bounded and $N \in \mathbb{N}$ be arbitrary. By the above remark we have that

$$\begin{aligned} \sup_{\varphi \in B'} \|f * \varphi\|_{L^1_{w_N}} &= \sup_{\varphi \in B'} \sup_{h \in B} \left| \int_{\mathbb{R}^d} f * \varphi(x)h(x)w_N(x)dx \right| \\ &= \sup_{\varphi \in B'} \sup_{h \in B} |\langle f, \check{\varphi} * (hw_N) \rangle|, \end{aligned}$$

for all $f \in (\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'$, where the last equality follows from Corollary 10.3.5(ii). Choose $\tilde{N} \geq N$ according to (10.3.1). It is now enough to notice that the set

$$\{\check{\varphi} * (hw_N) \mid \varphi \in B', h \in B\}$$

is bounded in $\dot{\mathcal{B}}_{1/w_{\tilde{N}}}(\mathbb{R}^d)$, as follows from (10.3.4) with $v = w_N$ and $w = w_{\tilde{N}}$. Next, we show that $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ is continuously included in $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_{bs}$. Let $\psi \in \mathcal{D}(\mathbb{R}^d)$. By Proposition 10.3.1 and (9.3.4) it suffices to show that

$$V_\psi^* : L^1_{\mathcal{W}}(\mathbb{R}^d_x) \hat{\otimes}_\varepsilon \mathcal{V}_{\text{pol}} C(\mathbb{R}^d_\xi) \rightarrow (\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_{bs}$$

is well-defined and continuous. Let $F \in L^1_{\mathcal{W}}(\mathbb{R}^d_x) \hat{\otimes}_\varepsilon \mathcal{V}_{\text{pol}} C(\mathbb{R}^d_\xi)$ be arbitrary. The linear functional

$$f : \dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d) \rightarrow \mathbb{C} : \varphi \rightarrow \int \int_{\mathbb{R}^{2d}} F(x, \xi) V_{\bar{\psi}} \varphi(x, -\xi) dx d\xi$$

is well-defined and continuous by Lemma 10.2.4. Since $V_\psi^* F = f|_{\mathcal{D}(\mathbb{R}^d)}$, we obtain that $V_\psi^* F \in (\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'$ and

$$\langle V_\psi^* F, \varphi \rangle = \int \int_{\mathbb{R}^{2d}} F(x, \xi) V_{\bar{\psi}} \varphi(x, -\xi) dx d\xi, \quad \varphi \in \dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d).$$

Finally, we show that V_ψ^* is continuous. Let $N \in \mathbb{N}$ and $B \subset \dot{\mathcal{B}}_{1/w_N}(\mathbb{R}^d)$ bounded be arbitrary. Choose $\tilde{N} \geq N$ according to (10.3.1). Lemma 10.2.4 implies that

$$\sup_{\varphi \in B} \sup_{(x, \xi) \in \mathbb{R}^{2d}} \frac{|V_{\bar{\psi}}\varphi(x, -\xi)|(1 + |\xi|)^n}{w_{\tilde{N}}(x)} < \infty$$

for all $n \in \mathbb{N}$. Hence, there is $v \in \overline{V}(\mathcal{V}_{\text{pol}})$ such that $|V_{\bar{\psi}}\varphi(x, -\xi)| \leq w_{\tilde{N}}(x)v(\xi)$, $(x, \xi) \in \mathbb{R}^{2d}$, for all $\varphi \in B$. Set $\bar{v} = v(\cdot)(1 + |\cdot|)^{d+1} \in \overline{V}(\mathcal{V}_{\text{pol}})$. Then,

$$\begin{aligned} \sup_{\varphi \in B} |\langle V_\psi^* F, \varphi \rangle| &\leq \sup_{\varphi \in B} \int \int_{\mathbb{R}^{2d}} |F(x, \xi)| |V_{\bar{\psi}}\varphi(x, -\xi)| dx d\xi \\ &\leq \int \int_{\mathbb{R}^{2d}} |F(x, \xi)| w_{\tilde{N}}(x) v(\xi) dx d\xi \\ &\leq \|F\|_{L_{w_{\tilde{N}}, \bar{v}}^1} \int_{\mathbb{R}^d} (1 + |\xi|)^{-(d+1)} d\xi \end{aligned}$$

for all $F \in L_{\mathcal{W}}^1(\mathbb{R}^d) \widehat{\otimes}_\varepsilon \mathcal{V}_{\text{pol}} C(\mathbb{R}_\xi^d)$. \square

In the sequel, we shall interchangeably use $\mathcal{O}'_C(\mathcal{D}, L_{\mathcal{W}}^1)$ and $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_{bs}$ depending on which point of view is most suitable for the given situation. We shall not explicitly refer to Theorem 10.3.3 when we do this. Finally, for later use, we point out the following corollary.

Corollary 10.3.6. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1). Let $\psi \in \mathcal{D}(\mathbb{R}^d) \setminus \{0\}$ and let $\gamma \in \mathcal{D}(\mathbb{R}^d)$ be a synthesis window for ψ . Then, the desingularization formula*

$$\langle f, \varphi \rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\bar{\gamma}} \varphi(x, -\xi) dx d\xi$$

holds for all $f \in (\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'$ and $\varphi \in \dot{\mathcal{B}}_{\mathcal{W}^\circ}$.

10.3.3 Topological properties

We now take a closer look at the locally convex structure of the space $\mathcal{O}'_C(\mathcal{D}, L_{\mathcal{W}}^1)$. We start with the ensuing technical lemma.

Lemma 10.3.7. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1). Then, we have the dense continuous inclusion $\mathcal{D}(\mathbb{R}^d) \hookrightarrow \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$.*

Proof. Notice that $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{D}_{L^1_{\mathcal{W}}}(\mathbb{R}^d) \subset \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ with continuous inclusions. We shall prove that both inclusions have dense range. We start by showing that $\mathcal{D}_{L^1_{\mathcal{W}}}(\mathbb{R}^d)$ is dense in $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$. Let $f \in \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ be arbitrary. Pick $\chi \in \mathcal{D}(\mathbb{R}^d)$ with $\text{supp } \chi \subseteq \overline{B}(0, 1)$ and $\int_{\mathbb{R}^d} \chi(x) dx = 1$. Set $\chi_k = k^d \chi(k \cdot)$ and $f_k = f * \chi_k \in \mathcal{D}_{L^1_{\mathcal{W}}}(\mathbb{R}^d)$ for $k \geq 1$. We claim that $f_k \rightarrow f$ in $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$. Let $B \subset \mathcal{D}(\mathbb{R}^d)$ bounded and $N \in \mathbb{N}$ be arbitrary. Choose $\tilde{N} \geq N$ according to (10.3.1). Hence,

$$\begin{aligned}
& \sup_{\varphi \in B} \|f_k * \varphi - f * \varphi\|_{L^1_{w_N}} \\
& \leq \sup_{\varphi \in B} \|f * \varphi * \chi_k - f * \varphi\|_{L^1_{w_N}} \\
& \leq \sup_{\varphi \in B} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\chi(t)| |f * \varphi(x - t/k) - f * \varphi(x)| dt w_N(x) dx \\
& \leq \frac{1}{k} \sup_{\varphi \in B} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\chi(t)| |t| \cdot \\
& \quad \sum_{|\beta|=1} \int_0^1 |\partial^\beta (f * \varphi)(x - \gamma t/k)| d\gamma dt w_N(x) dx \\
& \leq \frac{1}{k} \sum_{|\beta|=1} \sup_{\varphi \in B} \int_0^1 \int_{\overline{B}(0,1)} |\chi(t)| |t| h_{N, \tilde{N}}(\gamma t/k) \cdot \\
& \quad \int_{\mathbb{R}^d} |f * \partial^\beta \varphi(x - \gamma t/k)| w_{\tilde{N}}(x - \gamma t/k) dx d\gamma \\
& \leq \frac{C}{k},
\end{aligned}$$

where

$$C = \|\chi(t)t\|_{L^1, t} \|h_{N, \tilde{N}}\|_{L^\infty(\overline{B}(0,1))} \sum_{|\beta|=1} \sup_{\varphi \in B} \|f * \partial^\beta \varphi\|_{L^1_{w_{\tilde{N}}}} < \infty.$$

Next, we show that $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{D}_{L^1_{\mathcal{W}}}(\mathbb{R}^d)$. Let $\varphi \in \mathcal{D}_{L^1_{\mathcal{W}}}(\mathbb{R}^d)$ be arbitrary. Choose $\chi \in \mathcal{D}(\mathbb{R}^d)$ with $0 \leq \chi \leq 1$ and $\chi(0) = 1$.

Set $K = \text{supp } \chi$. Define $\chi_k = \chi(\cdot/k)$ and $\varphi_k = \chi_k \varphi \in \mathcal{D}(\mathbb{R}^d)$ for $k \geq 1$. We claim that $\varphi_k \rightarrow \varphi$ in $\mathcal{D}_{L^1_{\mathcal{W}}}(\mathbb{R}^d)$. For all $n, N \in \mathbb{N}$ it holds that

$$\begin{aligned}
& \|\varphi_k - \varphi\|_{\mathcal{D}_{L^1_{w_N}}^n} \\
&= \max_{|\alpha| \leq n} \|\partial^\alpha(\varphi(1 - \chi_k))\|_{L^1_{w_N}} \\
&\leq \max_{|\alpha| \leq n} \|(\partial^\alpha \varphi)(1 - \chi_k)\|_{L^1_{w_N}} \\
&+ \frac{1}{k} \max_{|\alpha| \leq n} \sum_{\beta \leq \alpha, \beta \neq 0} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta} \varphi(\partial^\beta \chi)(\cdot/k)\|_{L^1_{w_N}} \\
&\leq \max_{|\alpha| \leq n} \|(\partial^\alpha \varphi)(1 - \chi_k)\|_{L^1_{w_N}} + \frac{C}{k},
\end{aligned}$$

where $C = 2^n \|\varphi\|_{\mathcal{D}_{L^1_{w_N}}^n} \|\chi\|_{C^n(K)} < \infty$. We still need to show that $\|(\partial^\alpha \varphi)(1 - \chi_k)\|_{L^1_{w_N}} \rightarrow 0$ for all $|\alpha| \leq n$. Let $\varepsilon > 0$ be arbitrary. Choose $K' \Subset \mathbb{R}^d$ so large that

$$\int_{\mathbb{R}^d \setminus K'} |\partial^\alpha \varphi(x)| w_N(x) dx \leq \varepsilon.$$

Then,

$$\|(\partial^\alpha \varphi)(1 - \chi_k)\|_{L^1_{w_N}} \leq \|\partial^\alpha \varphi\|_{L^1_{w_N}} \sup_{x \in K'} (1 - \chi_k(x)) + \varepsilon.$$

The result now follows from the fact that $\chi_k \rightarrow 1$ uniformly on compact subsets of \mathbb{R}^d . \square

Proposition 10.3.8. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1).*

(i) *The canonical inclusion*

$$\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d) \rightarrow ((\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_{bs})'_b : \varphi \rightarrow (f \rightarrow \langle f, \varphi \rangle) \quad (10.3.5)$$

is a topological embedding.

(ii) Let $\psi \in \mathcal{D}(\mathbb{R}^d) \setminus \{0\}$ and let $\gamma \in \mathcal{D}(\mathbb{R}^d)$ be a synthesis window for ψ . Then, the mapping $\mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d) \rightarrow ((\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_{bs})'_b$ given by

$$\varphi \rightarrow \left(f \rightarrow \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\bar{\gamma}} \varphi(x, -\xi) dx d\xi \right) \quad (10.3.6)$$

is a continuous bijection whose restriction to $\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d)$ coincides with the inclusion (10.3.5).

Proof. (i) The *bs*-topology is coarser than the strong dual topology and finer than the weak-* topology on $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'$. Since $\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d)$ is barrelled (as it is an (LF) -space), a subset of $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'$ is therefore equicontinuous if and only if it is *bs*-bounded, which in turn yields that the mapping (10.3.5) is a strict morphism.

(ii) We start by showing that, for each $\varphi \in \mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d)$, the mapping

$$f \rightarrow \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\bar{\gamma}} \varphi(x, -\xi) dx d\xi$$

defines a continuous linear functional on $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$. Proposition 10.2.6 implies that there are $N \in \mathbb{N}$ and $v \in \overline{V}(\mathcal{V}_{\text{pol}})$ such that $|V_{\bar{\gamma}} \varphi(x, -\xi)| \leq w_N(x) v(\xi)$ for all $(x, \xi) \in \mathbb{R}^{2d}$. Set $\bar{v} = v(\cdot)(1 + |\cdot|)^{d+1} \in \overline{V}(\mathcal{V}_{\text{pol}})$. Then,

$$\begin{aligned} & \left| \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\bar{\gamma}} \varphi(x, -\xi) dx d\xi \right| \\ & \leq \frac{1}{|(\gamma, \psi)_{L^2}|} \int \int_{\mathbb{R}^{2d}} |V_\psi f(x, \xi)| w_N(x) v(\xi) dx d\xi \\ & \leq \frac{\|V_\psi f\|_{L^1_{w_N}, \bar{v}}}{|(\gamma, \psi)_{L^2}|} \int_{\mathbb{R}^d} (1 + |\xi|)^{-(d+1)} d\xi \end{aligned}$$

for all $f \in \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$. The claim now follows from Proposition 10.3.1. Next, we prove that the mapping (10.3.6) is continuous. It suffices to show that its restriction to $\mathcal{B}_{1/w_N}(\mathbb{R}^d)$ is continuous for each $N \in \mathbb{N}$. Choose $\tilde{N} \geq N$ according to (10.3.1). Now let

$B \subset \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ bounded be arbitrary. By Proposition 10.3.1 there is $n \in \mathbb{N}$ and $C > 0$ such that

$$\|V_\psi f(\cdot, \xi)\|_{L^1_{w_{\tilde{N}}}} \leq C(1 + |\xi|)^n, \quad \xi \in \mathbb{R}^d,$$

for all $f \in B$. Lemma 10.2.4 implies that there is $C' > 0$ such that

$$|V_{\tilde{\gamma}}\varphi(x, -\xi)| \leq \frac{C'\|\varphi\|_{\mathcal{B}^{n+d+1}_{1/w_N}}(x)}{(1 + |\xi|)^{n+d+1}}, \quad (x, \xi) \in \mathbb{R}^{2d},$$

for all $\varphi \in \mathcal{B}_{1/w_N}(\mathbb{R}^d)$. Hence,

$$\sup_{f \in B} \left| \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\tilde{\gamma}}\varphi(x, -\xi) dx d\xi \right| \leq C'' \|\varphi\|_{\mathcal{B}^{n+d+1}_{1/w_N}}$$

for all $\varphi \in \mathcal{B}_{1/w_N}(\mathbb{R}^d)$, where

$$C'' = \frac{CC'}{|(\gamma, \psi)_{L^2}|} \int_{\mathbb{R}^d} (1 + |\xi|)^{-(d+1)} < \infty.$$

Finally, we prove that (10.3.6) is surjective. Let $\Phi \in (\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}))'$ be arbitrary. Denote by $\iota : \mathcal{D}(\mathbb{R}^d) \rightarrow \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ the canonical inclusion and set $g = \Phi \circ \iota \in \mathcal{D}'(\mathbb{R}^d)$. By (9.3.5) it holds that

$$\Phi(\iota(\chi)) = \langle g, \chi \rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi \chi(x, \xi) V_{\tilde{\gamma}} g(x, -\xi) dx d\xi$$

for all $\chi \in \mathcal{D}(\mathbb{R}^d)$. By Lemma 10.3.7 it therefore suffices to show that $g \in \mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d)$ or, equivalently, that $V_\theta g \in \mathcal{W}^\circ \otimes_i \mathcal{W}_{\text{pol}} C(\mathbb{R}^{2d}_{x, \xi})$, where $\theta \in \mathcal{D}(\mathbb{R}^d)$ (Proposition 10.2.6 and (9.3.4)). Since Φ is continuous, there is $N \in \mathbb{N}$ and a bounded set $B \subset \dot{\mathcal{B}}_{w_N}(\mathbb{R}^d)$ such that

$$\begin{aligned} |V_\theta g(x, \xi)| &= |\Phi(\iota(\overline{M_\xi T_x \theta}))| \\ &\leq \sup_{\varphi \in B} \left| \int_{\mathbb{R}^d} \varphi(t) \overline{M_\xi T_x \theta(t)} dt \right| = \sup_{\varphi \in B} |V_\theta \varphi(x, \xi)|. \end{aligned}$$

The required bounds for $|V_\theta g|$ therefore directly follow from Lemma 10.2.4. The last statement is a reformulation of Corollary 10.3.6. \square

Since $\mathcal{D}(\mathbb{R}^d)$ is bornological, the last part of the proof of Proposition 10.3.8(ii) implies the following corollary.

Corollary 10.3.9. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1). Then, every bounded linear functional on $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ is continuous.*

Corollary 10.3.10. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1) and (9.1.2). Then,*

$$\mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d) \rightarrow ((\mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_{bs})'_b : \varphi \rightarrow (f \rightarrow \langle f, \varphi \rangle)$$

is a topological isomorphism.

We believe that the mapping (10.3.6) might *always* be a topological isomorphism but we are not able to prove this in general. However, we now show that this is indeed the case if \mathcal{W}° satisfies (Ω) . As it will turn out, this suffices for our main purpose.

Proposition 10.3.11. *Let \mathcal{W} be an increasing weight system satisfying (10.3.1) such that \mathcal{W}° satisfies (Ω) . Then, (10.3.6) is a topological isomorphism.*

Proof. We still need to show that the mapping (10.3.6) is open. Let $\chi \in \mathcal{D}(\mathbb{R}^d)$ be some non-zero window function. By Corollaries 10.2.8 and 9.3.6 it suffices to show that for every $u \in \overline{U}(\mathcal{W}^\circ \otimes_i \mathcal{W}_{\text{pol}})$ there is a set $B \subset \mathcal{E}'(\mathbb{R}^d)$ that is bounded in $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_{bs}$ such that $\|V_\chi \varphi\|_{Cu} \leq \sup_{f \in B} |\langle f, \varphi \rangle|$ for all $\varphi \in \mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d)$. Define

$$B = \{(\overline{M_\xi T_x \chi})u(x, \xi) \mid (x, \xi) \in \mathbb{R}^{2d}\} \subset \mathcal{E}'(\mathbb{R}^d)$$

and observe that B satisfies all requirements because

$$\begin{aligned} \|V_\chi \varphi\|_{Cu} &= \sup_{(x, \xi) \in \mathbb{R}^{2d}} |V_\chi \varphi(x, \xi)| u(x, \xi) \\ &= \sup_{(x, \xi) \in \mathbb{R}^{2d}} |\langle \overline{M_\xi T_x \chi}, \varphi \rangle| u(x, \xi) = \sup_{f \in B} |\langle f, \varphi \rangle| \end{aligned}$$

for all $\varphi \in \mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d)$. □

We have all necessary ingredients to prove the main theorem of this section.

Theorem 10.3.12. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1). Then, the following statements are equivalent:*

- (i) \mathcal{W}° satisfies (Ω) .
- (ii) $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ is ultrabornological.
- (iii) $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_b = \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$.

Proof. A l.c.s. is said to be infrabarrelled if every strongly bounded set in its dual is equicontinuous. Next, a l.c.s. E is bornological if and only if its infrabarrelled and every bounded linear functional on E is continuous³. Every complete bornological l.c.s. is ultrabornological.

(i) \Rightarrow (ii) Since $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ is complete (Corollary 10.3.2), we only need to prove that it is bornological. We already pointed out that every bounded linear functional on $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ is continuous (Corollary 10.3.9). We now show that $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ is infrabarrelled. Choose $\psi, \gamma \in \mathcal{D}(\mathbb{R}^d)$ with $(\gamma, \psi)_{L^2} = 1$. By Propositions 10.3.11 and 10.3.1 it suffices to show that for every bounded set $B \subset \mathcal{B}_{\mathcal{W}}(\mathbb{R}^d)$ there are $v \in \overline{V}(\mathcal{V}_{\text{pol}})$, $N \in \mathbb{N}$, and $C > 0$ such that

$$\sup_{\varphi \in B} \left| \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\bar{\gamma}}(x, -\xi) dx d\xi \right| \leq C \|V_\psi f\|_{L^1_{w_N, v}}$$

for all $f \in \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$. Since $\mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d)$ is regular (Theorem 10.2.1), there is $N \in \mathbb{N}$ such that B is contained and bounded in $\mathcal{B}_{1/w_N}(\mathbb{R}^d)$. Choose $\tilde{N} \geq N$ according to (10.3.1). Lemma 10.2.4 implies that

$$\sup_{\varphi \in B} \sup_{(x, \xi) \in \mathbb{R}^{2d}} \frac{|V_{\bar{\gamma}} \varphi(x, -\xi)| (1 + |\xi|)^n}{w_{\tilde{N}}(x)} < \infty$$

³In fact, for E to be bornological it suffices that it is Mackey (= every convex weak-* compact set in its dual is equicontinuous) and every bounded linear functional on E is continuous.

for all $n \in \mathbb{N}$. Hence, there is $v \in \overline{V}(\mathcal{V}_{\text{pol}})$ such that $|V_{\bar{\gamma}}\varphi(x, -\xi)| \leq w_{\bar{N}}(x)v(\xi)$, $(x, \xi) \in \mathbb{R}^{2d}$, for all $\varphi \in B$. Set $\bar{v} = v(\cdot)(1 + |\cdot|)^{d+1} \in \overline{V}(\mathcal{V}_{\text{pol}})$. We obtain that

$$\begin{aligned} & \sup_{\varphi \in B} \left| \int \int_{\mathbb{R}^{2d}} V_{\psi} f(x, \xi) V_{\bar{\gamma}} \varphi(x, -\xi) dx d\xi \right| \\ & \leq \|V_{\psi} f\|_{L^1_{w_{\bar{N}}, \bar{v}}} \int_{\mathbb{R}^d} (1 + |\xi|)^{-(d+1)} d\xi \end{aligned}$$

for all $f \in \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$.

(ii) \Rightarrow (iii) It suffices to show that the bs -topology is finer than the strong dual topology on $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'$. Let $U \subset (\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_b$ be an arbitrary neighbourhood of 0. We may assume that $U = B^\circ$, where B° the polar set of some bounded set B in $\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d)$. Since (10.3.5) is a topological embedding and $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_{bs}$ is ultrabornological, there is $N \in \mathbb{N}$ and a bounded set $B' \subset \dot{\mathcal{B}}_{1/w_N}(\mathbb{R}^d)$ such that $(B')^\circ \subseteq U$.

(iii) \Rightarrow (i) By Theorem 10.2.1 it suffices to show that $\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d)$ is β -regular. Let $N \in \mathbb{N}$ be arbitrary and let B be a subset of $\dot{\mathcal{B}}_{1/w_N}(\mathbb{R}^d)$ that is bounded in $\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d)$. Our assumption implies that there is $M \in \mathbb{N}$ and a bounded set $B' \subset \dot{\mathcal{B}}_{1/w_M}(\mathbb{R}^d)$ such that $(B')^\circ \subseteq B^\circ$, where the polarity is taken twice with respect to the dual system $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d), (\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))')$. We may assume that $M \geq N$. We now show that B is bounded in $\dot{\mathcal{B}}_{1/w_M}(\mathbb{R}^d)$. Let $n \in \mathbb{N}$ be arbitrary and choose $C > 0$ such that $\|\varphi\|_{\mathcal{B}^n_{1/w_M}} \leq C$ for all $\varphi \in B'$. Hence,

$$f_{\alpha, x} := \frac{(-1)^{|\alpha|} \partial^\alpha (T_x \delta)}{C w_M(x)} \in (B')^\circ \subseteq B^\circ$$

for all $x \in \mathbb{R}^d$ and $|\alpha| \leq n$. We obtain that

$$\sup_{\varphi \in B} \|\varphi\|_{\mathcal{B}^n_{1/w_M}} = C \sup_{\varphi \in B} \max_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} |\langle f_{\alpha, x}, \varphi \rangle| \leq C.$$

□

10.4 Applications

We now apply the results from Section 10.3 to study weighted \mathcal{D}'_{L^1} spaces and the convolutor spaces of the Gelfand-Shilov spaces $\mathcal{K}\{M_p\}$ [64, Chap. II].

10.4.1 Weighted $\mathcal{D}'_{L^1}(\mathbb{R}^d)$ spaces

Let ω be a positive measurable function. We denote by $\mathcal{D}'_{L^1_\omega}(\mathbb{R}^d)$ the strong dual of the Fréchet space $\dot{\mathcal{B}}_{1/\omega}(\mathbb{R}^d)$.

Theorem 10.4.1. *Let ω be a positive measurable function such that*

$$g = \operatorname{ess\,sup}_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L^\infty_{\operatorname{loc}}(\mathbb{R}^d). \quad (10.4.1)$$

Then, the following properties hold:

- (i) $\mathcal{D}'_{L^1_\omega}(\mathbb{R}^d) = \mathcal{O}'_C(\mathcal{D}, L^1_\omega)$ as locally convex spaces.
- (ii) *The strong dual of $\mathcal{D}'_{L^1_\omega}(\mathbb{R}^d)$ is isomorphic to $\mathcal{B}_{1/\omega}(\mathbb{R}^d)$. More precisely, let $\psi \in \mathcal{D}(\mathbb{R}^d) \setminus \{0\}$ and let $\gamma \in \mathcal{D}(\mathbb{R}^d)$ be a synthesis window for ψ . Then, the mapping $\mathcal{B}_{1/\omega}(\mathbb{R}^d) \rightarrow (\mathcal{D}'_{L^1_\omega}(\mathbb{R}^d))'_b$ given by*

$$\varphi \rightarrow \left(f \rightarrow \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\bar{\gamma}} \varphi(x, -\xi) dx d\xi \right)$$

is a topological isomorphism.

- (iii) $\dot{\mathcal{B}}_{1/\omega}(\mathbb{R}^d)$ is a distinguished Fréchet space.

Proof. We may assume that ω is continuous. Indeed, for otherwise consider the continuous weight $\tilde{\omega} = \omega * \varphi$, where $\varphi \in \mathcal{D}(\mathbb{R}^d)$ is non-negative and satisfies $\int_{\mathbb{R}^d} \varphi(t) dt = 1$. Set $K = \operatorname{supp} \varphi$. Then,

$$\|g\|_{L^\infty(K)}^{-1} \omega(x) \leq \tilde{\omega}(x) \leq \|\check{g}\|_{L^\infty(K)} \omega(x), \quad x \in \mathbb{R}^d.$$

Hence, $\mathcal{B}_{1/\omega}(\mathbb{R}^d) = \mathcal{B}_{1/\tilde{\omega}}(\mathbb{R}^d)$, $\dot{\mathcal{B}}_{1/\omega}(\mathbb{R}^d) = \dot{\mathcal{B}}_{1/\tilde{\omega}}(\mathbb{R}^d)$, and $L_{\omega}^1(\mathbb{R}^d) = L_{\tilde{\omega}}^1(\mathbb{R}^d)$. Moreover,

$$\tilde{\omega}(x+t) = \int_{\mathbb{R}^d} \omega(x+t-y)\varphi(y)dy \leq g(t)\tilde{\omega}(x), \quad x, t \in \mathbb{R}^d,$$

whence

$$\sup_{x \in \mathbb{R}^d} \frac{\tilde{\omega}(x + \cdot)}{\tilde{\omega}(x)} \in L_{\text{loc}}^{\infty}(\mathbb{R}^d).$$

From now on we assume that ω is continuous. Set $\mathcal{W} = (\omega)_N$ and notice that \mathcal{W}° satisfies (Ω) . Properties (i)-(iii) therefore follow immediately from Proposition 10.3.11, Theorem 10.3.12, and the fact that a Fréchet space is distinguished if and only if its strong dual is infrabarrelled [97, p. 400 (3)]. \square

Remark 10.4.2. Observe that the function g from (10.4.1) is submultiplicative. Since every locally bounded submultiplicative function is exponentially bounded, the weight ω satisfies (10.4.1) if and only if

$$\omega(x+y) \leq C\omega(x)e^{\lambda|y|}, \quad x, y \in \mathbb{R}^d,$$

for some $C, \lambda > 0$. Interestingly, (10.4.1) is also equivalent to the fact that the space $L_{\omega}^1(\mathbb{R}^d)$ is translation-invariant, that is, $T_x(L_{\omega}^1(\mathbb{R}^d)) \subseteq L_{\omega}^1(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$ (cf. [51, Prop. 10]).

10.4.2 Convolution spaces of Gelfand-Shilov spaces

Let $\mathcal{W} = (w_N)_N$ be an increasing weight system. We define the Fréchet spaces

$$\mathcal{B}_{\mathcal{W}}(\mathbb{R}^d) := \varprojlim_{N \in \mathbb{N}} \mathcal{B}_{w_N}(\mathbb{R}^d), \quad \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d) := \varprojlim_{N \in \mathbb{N}} \dot{\mathcal{B}}_{w_N}(\mathbb{R}^d).$$

If \mathcal{W} satisfies (9.1.2), then $\mathcal{B}_{\mathcal{W}}(\mathbb{R}^d) = \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d)$. The corresponding convolution spaces are defined as follows. Set $\check{\mathcal{W}} = (\check{w}_N)_N$ and

$$\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}}) := \{f \in (\dot{\mathcal{B}}_{\check{\mathcal{W}}}(\mathbb{R}^d))' \mid f * \varphi \in \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d) \text{ for all } \varphi \in \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d)\}.$$

We endow $\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}})$ with the initial topology with respect to the mapping

$$\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}}) \rightarrow L_b(\dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d), \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi).$$

The goal of this subsection is to study the structural and topological properties of $\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}})$. We shall need to impose the following conditions on \mathcal{W} :

$$\forall N \exists M > N : w_N/w_M \in L^1(\mathbb{R}^d) \quad (10.4.2)$$

and

$$\begin{aligned} \forall N \exists M, K \geq N \exists C > 0 \forall x, y \in \mathbb{R}^d : \\ w_N(x + y) \leq C w_M(x) w_K(y). \end{aligned} \quad (10.4.3)$$

Clearly, (10.4.2) implies (10.3.1). Moreover, it is worth mentioning that Gelfand and Shilov used condition (10.4.2) for deriving structural theorems for $(\mathcal{B}_{\mathcal{W}}(\mathbb{R}^d))'$ [64, p. 113]. The following observation is fundamental.

Proposition 10.4.3. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.4.2) and (10.4.2). Then, $\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}}) = \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ as locally convex spaces.*

In order to be able to show Proposition 10.4.3, we must first study the Fréchet spaces $\mathcal{B}_{\mathcal{W}}(\mathbb{R}^d)$ and $\dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d)$ in some more detail. Lemmas 10.2.4 and 10.2.5 immediately imply the following two results.

Proposition 10.4.4. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1). Then,*

$$V_{\psi} : \mathcal{B}_{\mathcal{W}}(\mathbb{R}^d) \rightarrow \mathcal{W} \otimes \mathcal{W}_{\text{pol}} C(\mathbb{R}^{2d}_{x,\xi})$$

and

$$V_{\psi}^* : \mathcal{W} \otimes \mathcal{W}_{\text{pol}} C(\mathbb{R}^{2d}_{x,\xi}) \rightarrow \mathcal{B}_{\mathcal{W}}(\mathbb{R}^d)$$

are well-defined continuous mappings.

Proposition 10.4.5. *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1). Then,*

$$V_\psi : \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d) \rightarrow (\mathcal{W} \otimes \mathcal{W}_{\text{pol}})_0 C(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi^* : (\mathcal{W} \otimes \mathcal{W}_{\text{pol}})_0 C(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d)$$

are well-defined continuous mappings.

We obtain the following interesting corollary; it is the weighted analogue of a classical result of Schwartz [151, p. 200].

Corollary 10.4.6. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.3.1). Then, $\mathcal{D}_{L_{\mathcal{W}}^1}(\mathbb{R}^d) \subseteq \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d)$ with continuous inclusion. If, in addition, \mathcal{W} satisfies (10.4.2), then $\mathcal{D}_{L_{\mathcal{W}}^1}(\mathbb{R}^d) = \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d) = \mathcal{B}_{\mathcal{W}}(\mathbb{R}^d)$.*

Proof. We start by showing that $\mathcal{D}_{L_{\mathcal{W}}^1}(\mathbb{R}^d)$ is continuously included in $\dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d)$. By (9.3.4) and Proposition 10.4.5 it suffices to show that

$$V_\psi : \mathcal{D}_{L_{\mathcal{W}}^1}(\mathbb{R}^d) \rightarrow (\mathcal{W} \otimes \mathcal{W}_{\text{pol}})_0 C(\mathbb{R}_{x,\xi}^{2d})$$

is well-defined and continuous. This can be done by modifying the proof of Lemma 10.2.4. The second part follows from the fact that (10.4.2) implies that $\mathcal{B}_{\mathcal{W}}(\mathbb{R}^d)$ is continuously included in $\mathcal{D}_{L_{\mathcal{W}}^1}(\mathbb{R}^d)$. \square

Proof of Proposition 10.4.3. The continuous inclusion $\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}}) \subseteq \mathcal{O}'_C(\mathcal{D}, L_{\mathcal{W}}^1)$ is clear from (10.4.2). For the converse inclusion, we introduce the ensuing space

$$\begin{aligned} \mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}}, L_{\mathcal{W}}^1) := \{f \in (\dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d))' \mid f * \varphi \in L_{\mathcal{W}}^1(\mathbb{R}^d) \\ \text{for all } \varphi \in \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d)\}. \end{aligned}$$

Similarly as in Corollary 10.3.5 and in the first part of the proof of Theorem 10.3.3, but by using (10.4.2) instead of (10.3.1), one can

show that $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_{bs}$ is continuously included in $\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}}, L^1_{\mathcal{W}})$. Next, we notice that $f \in (\dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d))'$ belongs to $\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}}, L^1_{\mathcal{W}})$ if and only if $f * \varphi \in \mathcal{D}_{L^1_{\mathcal{W}}}(\mathbb{R}^d)$ for all $\varphi \in \dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d)$ and that the topology of $\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}}, L^1_{\mathcal{W}})$ coincides with the initial topology with respect to the mapping

$$\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}}, L^1_{\mathcal{W}}) \rightarrow L_b(\dot{\mathcal{B}}_{\mathcal{W}}(\mathbb{R}^d), \mathcal{D}_{L^1_{\mathcal{W}}}(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi).$$

Hence, the result follows from Corollary 10.4.6. \square

Proposition 10.4.3 and Theorem 10.3.12 imply the following important result.

Theorem 10.4.7. *Let $\mathcal{W} = (w_N)_N$ be an increasing weight system satisfying (10.4.2) and (10.4.2). Then, the following statements are equivalent:*

- (i) \mathcal{W}° satisfies (Ω) .
- (ii) $\mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}})$ is ultrabornological.
- (iii) $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'_b = \mathcal{O}'_C(\dot{\mathcal{B}}_{\mathcal{W}})$.

Finally, we apply Theorem 10.4.7 to study the convolutor spaces of several Gelfand-Shilov spaces that are frequently used in the literature. For this, we evaluate conditions (10.4.2), (10.4.2), and (Ω) for two classes of increasing weight systems. Namely, let ω be a positive continuous increasing function on $[0, \infty)$ and extend ω to \mathbb{R}^d via $\omega(x) = \omega(|x|)$, $x \in \mathbb{R}^d$. We define the following increasing weight systems on \mathbb{R}^d :

$$\mathcal{W}_\omega := (e^{N\omega})_N, \quad \widetilde{\mathcal{W}}_\omega := (e^{\omega(N \cdot)})_N.$$

The function ω is said to satisfy (α) (cf. [21]) if there are $C'_0, H' \geq 1$ such that

$$\omega(2t) \leq H'\omega(t) + \log C'_0, \quad t \geq 0.$$

Proposition 10.4.8. *Let ω be a positive continuous increasing function on $[0, \infty)$ with $\log(1+t) = O(\omega(t))$ satisfying (α) . Then, \mathcal{W}_ω satisfies (10.4.2) and (10.4.2) while \mathcal{W}_ω° satisfies (Ω) . Consequently, $\mathcal{O}'_C(\mathcal{B}_{\mathcal{W}_\omega})$ is ultrabornological.*

Proof. The proof is simple and therefore omitted. \square

Example 10.4.9. For $\omega(t) = \log(1+t)$ we have that $\mathcal{B}_{\mathcal{W}_\omega}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$. Proposition 10.4.8 implies that the space of convolutors $\mathcal{O}'_C(\mathcal{S}) = \mathcal{O}'_C(\mathbb{R}^d)$ is ultrabornological. This fact was first shown by Grothendieck [73, Chap. II, Thm. 16, p. 131]. He showed that $\mathcal{O}'_C(\mathbb{R}^d)$ is isomorphic to a complemented subspace of $s \hat{\otimes} s'$ and proved explicitly that the latter space is ultrabornological.

Example 10.4.10. For $\omega(t) = t$ we have that $\mathcal{B}_{\mathcal{W}_\omega}(\mathbb{R}^d) = \mathcal{K}_1(\mathbb{R}^d)$, the space of exponentially rapidly decreasing smooth functions [75]. Proposition 10.4.8 yields that the space of convolutors $\mathcal{O}'_C(\mathcal{K}_1)$ is ultrabornological. This fact was claimed by Zielezny in [166] but his argument seems to contain a gap (in particular, the proof of [166, Thm. 9] does not seem to be correct).

Example 10.4.11. More generally, let $\omega(t) = t^p$, $p > 0$, and set $\mathcal{B}_{\mathcal{W}_\omega}(\mathbb{R}^d) = \mathcal{K}_p(\mathbb{R}^d)$. For $p > 1$ the convolutor spaces $\mathcal{O}'_C(\mathcal{K}_p)$ were studied in [144], where the hypoelliptic convolution operators in $(\mathcal{K}_p(\mathbb{R}^d))'$ are characterized in terms of their Fourier transform. However, the topological properties of $\mathcal{O}'_C(\mathcal{K}_p)$ do not seem to have been studied yet. Proposition 10.4.8 implies that the space of convolutors $\mathcal{O}'_C(\mathcal{K}_p)$ is ultrabornological for each $p > 0$.

Proposition 10.4.12. *Let ω be a positive continuous increasing function on $[0, \infty)$ with $\log(1+t) = O(\omega(t))$ such that*

$$2\omega(t) \leq \omega(Ht) + \log C_0, \quad t \geq 0, \quad (10.4.4)$$

for some $C_0, H \geq 1$. Then, $\widetilde{\mathcal{W}}_\omega$ satisfies (10.4.2) and (10.4.2) while $\widetilde{\mathcal{W}}_\omega^\circ$ satisfies (Ω) if and only if ω satisfies (α) . Consequently, $\mathcal{O}'_C(\mathcal{B}_{\mathcal{W}_\omega})$ is ultrabornological if and only if ω satisfies (α) .

Proof. By iterating (10.4.4) we obtain that for every $\lambda > 0$ there are $L, B \geq 1$ such that $\lambda\omega(t) \leq \omega(Lt) + B$ for all $t \geq 0$. Condition (10.4.2) therefore follows from the assumption $\log(1+t) = O(\omega(t))$. Next, (10.4.2) is a consequence of the fact that ω is increasing. We now show that $\widetilde{\mathcal{W}}_\omega^\circ$ satisfies (Ω) if and only if ω satisfies (α) . For this, observe that $\widetilde{\mathcal{W}}_\omega^\circ$ satisfies (Ω) if and only if there is $M \geq 1$ such that for all $K \geq M$ there are $C, S \geq 1$ such that

$$\omega(Kt) - \omega(t) \leq C(\omega(Mt) - \omega(t)) + S, \quad t \geq 0. \quad (10.4.5)$$

This shows that ω satisfies (α) if $\widetilde{\mathcal{W}}_\omega^\circ$ satisfies (Ω) . Conversely, suppose that ω satisfies (α) . We shall show (10.4.5) for $M = H$, where H is chosen according to (10.4.4). Let $K \geq H$ be arbitrary. By iterating (α) we obtain that there are $L, B \geq 1$ such that $\omega(Kt) \leq L\omega(t) + B$ for all $t \geq 0$. Hence,

$$\begin{aligned} \omega(Kt) - \omega(t) &\leq (L-1)\omega(t) + B \\ &\leq (L-1)(\omega(Ht) - \omega(t)) + (L-1)\log C_0 + B. \end{aligned}$$

□

Remark 10.4.13. Since condition (α) implies that ω is polynomially bounded, Proposition 10.4.12 in particular yields that $\mathcal{O}'_C(\mathcal{B}_{\widetilde{\mathcal{W}}_\omega})$ is not ultrabornological for any weight function ω satisfying (10.4.4) and $O(t^\sigma) = \omega(t)$ for all $\sigma > 0$. Concrete examples are given by: $\omega(t) = e^{t^\sigma \log(1+t)^\tau}$ with $\sigma > 0$ and $\tau \geq 0$ or $\sigma = 0$ and $\tau > 1$. On the other hand, there are also polynomially bounded weights ω for which $\mathcal{O}'_C(\mathcal{B}_{\widetilde{\mathcal{W}}_\omega})$ fails to be ultrabornological, as is shown in the following example.

Example 10.4.14. *For any $\sigma > 0$ there is a weight function ω with $\log(1+t) = O(\omega(t))$ and $\omega(t) = O(t^\sigma)$ such that ω satisfies (10.4.4) but violates (α) .* In [103, Example 3.3], Langenbruch constructed a weight sequence M_p satisfying $(M.1)$, $(M.2)$, and $(M.3)'$ but not $(M.2)^*$. Hence, its associated function M is continuous, $\log(1 +$

$t) = o(M(t))$, $M(t) = o(t)$, and M satisfies (10.4.4) but violates (α) (cf. Subsection 2.2.1). Therefore, $\omega(t) = M(t^\sigma)$ satisfies all requirements.

We now further specialize our results to a class of weights introduced by Gelfand and Shilov in [64, Chap. IV, Appendix 2]⁴. Let μ be an increasing positive function on $[0, \infty)$ and consider $\omega(t) = \int_0^t \mu(s)ds$, $t \geq 0$. The spaces $\mathcal{B}_{\widetilde{\omega}}(\mathbb{R}^d) = \mathcal{K}_\omega(\mathbb{R}^d)$ where studied by Abdullah [2].

Theorem 10.4.15. *Let μ be an increasing positive function on $[0, \infty)$ and set $\omega(t) = \int_0^t \mu(s)ds$. Then, the following statements are equivalent:*

- (i) $\mathcal{O}'_C(\mathcal{K}_\omega)$ is ultrabornological.
- (ii) ω satisfies (α) .
- (iii) There are $L, B \geq 1$ such that

$$\mu(2s) \leq L\mu(s) + \frac{B}{s}, \quad s > 0. \quad (10.4.6)$$

Proof. Clearly, $\log(1+t) = O(\omega(t))$ (in fact, $t = O(\omega(t))$). Next, we show that ω satisfies (10.4.4). Since μ is increasing, we obtain that

$$\begin{aligned} 2\omega(t) &\leq \int_0^t \mu(s)ds + \int_0^t \mu(s+t)ds = \int_0^t \mu(s)ds + \int_t^{2t} \mu(s)ds \\ &= \int_0^{2t} \mu(s)ds = \omega(2t). \end{aligned}$$

Hence, in view of Proposition 10.4.12, it suffices to show that ω satisfies (α) if and only if μ satisfies (10.4.6). Suppose first that μ

⁴The function $\int_0^t \mu(s)ds$ is there denoted by M instead of ω and the authors impose that μ is continuous and $\mu(s) \rightarrow \infty$.

satisfies (10.4.6). Then,

$$\begin{aligned}
\omega(2t) - \omega(t) &= \int_t^{2t} \mu(s) ds = 2 \int_{t/2}^t \mu(2s) ds \\
&\leq 2 \int_{t/2}^t \left(L\mu(s) + \frac{B}{s} \right) ds \\
&\leq 2L \int_0^t \mu(s) ds + 2(\log 2)B \\
&= 2L\omega(t) + 2(\log 2)B.
\end{aligned}$$

Conversely, assume that ω satisfies (α) . By applying (α) twice, we obtain that

$$\omega(4s) - \omega(2s) \leq (H' - 1)\omega(2s) + \log C'_0 \leq H'(H' - 1)\omega(s) + H' \log C'_0$$

Since μ is increasing, we have that

$$\begin{aligned}
\mu(2s) &= \frac{\mu(2s)}{\log 2} \int_{2s}^{4s} \frac{1}{u} du \leq \frac{1}{\log 2} \int_{2s}^{4s} \frac{\mu(u)}{u} du \\
&\leq \frac{1}{2(\log 2)s} \int_{2s}^{4s} \mu(u) du = \frac{1}{2(\log 2)s} (\omega(4s) - \omega(2s)) \\
&\leq \frac{H'(H' - 1)}{2(\log 2)s} \omega(s) + \frac{H' \log C'_0}{2(\log 2)s} = \frac{H'(H' - 1)}{2(\log 2)s} \int_0^s \mu(u) du \\
&\quad + \frac{H' \log C'_0}{2(\log 2)s} \leq \frac{H'(H' - 1)\mu(s)}{2(\log 2)} + \frac{H' \log C'_0}{2(\log 2)s}.
\end{aligned}$$

□

Remark 10.4.16. The topological properties of the spaces $\mathcal{O}'_C(\mathcal{K}_\omega)$ were studied by Abdullah in [1, 3]. On [3, p. 179] he states that the spaces $\mathcal{O}'_C(\mathcal{K}_\omega)$ are *always* ultrabornological but he does not provide a proof of this assertion. Theorem 10.4.15 shows that his claim is false. By Remark 10.4.13 the space $\mathcal{O}'_C(\mathcal{K}_\omega)$ is not ultrabornological if $O(t^\sigma) = \omega(t)$ for all $\sigma > 0$. On the other hand, one can use Example 10.4.14 to construct weights ω of the form $\omega(t) = \int_0^t \mu(s) ds$ with $\omega(t) = O(t^{1+\sigma})$, for a fixed but arbitrary $\sigma > 0$, such that ω does not satisfy (α) .

Chapter 11

The ultradifferentiable case

11.1 Introduction

We now introduce and study two classes of weighted convolutor spaces in the setting of tempered ultradistributions defined via increasing and decreasing weight systems, respectively. Our main goal is to make an analysis of these spaces similar to the one that has been made for $\mathcal{O}'_C(\mathcal{D}, L^1_W)$ in Chapter 10. Most importantly, we determine a topological predual of these spaces and give necessary and sufficient conditions for them to be ultrabornological in terms of their defining weight systems. Our results apply to the important case of spaces of convolutors of the Gelfand-Shilov spaces $\mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ and $\mathcal{S}^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d)$. Such convolutor spaces have already been considered in [52, 50] but their topological properties do not seem to have been thoroughly investigated yet (cf. the question posed after [50, Thm. 3.3]).

As often happens in ultradistribution theory, it will be conceptually rather easy to extend our techniques employed in the smooth case (Chapter 10) to the Beurling case while several new ideas will be used in the Roumieu case. Most notably, we provide a projective description of a class of weighted (LB) -spaces of ultradifferentiable

functions of Roumieu type. This description shall be essential to obtain the desired results but is also of independent interest. Our arguments are based on the mapping properties of the STFT and the projective description of weighted (LB) -spaces of continuous functions (more precisely, Theorem 9.1.3). This method has the advantage that one can work under very mild conditions and that it avoids duality theory; in fact, our result can be employed to more easily study dual spaces, e.g. one might deduce structural theorems from it without resorting to a rather complicated dual Mittag-Leffler argument (cf. [89, 132]).

This chapter is organized as follows. Several weighted spaces of ultradifferentiable functions are defined and investigated in Section 11.2. We introduce the weighted convolutor spaces we shall be concerned with in Section 11.3 and study various of their structural and topological properties via the STFT there. Finally, in Section 11.4, we specialize our results to the Gelfand-Shilov spaces $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$.

11.2 Weighted inductive limits of spaces of ultradifferentiable functions

In this section, we study various weighted spaces of ultradifferentiable functions. More precisely, we discuss their locally convex structure and establish the mapping properties of the STFT on these spaces. Based upon these mapping properties, we give a projective description of a class of weighted (LB) -spaces of ultradifferentiable functions of Roumieu type.

11.2.1 Definition and basic properties

Let v be a non-negative function on \mathbb{R}^d and let M_p be a weight sequence. For $h > 0$ we write $\mathcal{B}_v^{M_p, h}(\mathbb{R}^d)$ for the seminormed space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{B}_v^{M_p, h}} := \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)| v(x)}{h^{|\alpha|} M_\alpha} < \infty.$$

If v is positive, then $\|\cdot\|_{\mathcal{B}_v^{M_p, h}}$ is actually a norm and if, in addition, $1/v$ is locally bounded, then $\mathcal{B}_v^{M_p, h}(\mathbb{R}^d)$ is complete and, thus, a Banach space. We set

$$\mathcal{B}_v^{(M_p)}(\mathbb{R}^d) := \varprojlim_{h \rightarrow 0^+} \mathcal{B}_v^{M_p, h}(\mathbb{R}^d), \quad \mathcal{B}_v^{\{M_p\}}(\mathbb{R}^d) := \varinjlim_{h \rightarrow \infty} \mathcal{B}_v^{M_p, h}(\mathbb{R}^d).$$

Let $\mathcal{W} = (w_n)_n$ be an increasing weight system and let $\mathcal{V} = (v_N)_N$ be a decreasing weight system. For $h > 0$ we define

$$\mathcal{B}_{\mathcal{W}}^{M_p, h}(\mathbb{R}^d) := \varprojlim_{n \in \mathbb{N}} \mathcal{B}_{w_n}^{M_p, h}(\mathbb{R}^d), \quad \mathcal{B}_{\mathcal{V}}^{M_p, h}(\mathbb{R}^d) := \varinjlim_{N \in \mathbb{N}} \mathcal{B}_{v_N}^{M_p, h}(\mathbb{R}^d).$$

We shall mainly be interested in the ensuing (LF) -spaces

$$\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d) := \varinjlim_{h \rightarrow \infty} \mathcal{B}_{\mathcal{W}}^{M_p, h}(\mathbb{R}^d), \quad \mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d) := \varprojlim_{N \in \mathbb{N}} \mathcal{B}_{v_N}^{(M_p)}(\mathbb{R}^d).$$

However, we shall also consider the spaces

$$\mathcal{B}_{\mathcal{W}}^{(M_p)}(\mathbb{R}^d) := \varprojlim_{h \rightarrow 0^+} \mathcal{B}_{\mathcal{W}}^{M_p, h}(\mathbb{R}^d), \quad \mathcal{B}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d) := \varinjlim_{N \in \mathbb{N}} \mathcal{B}_{v_N}^{\{M_p\}}(\mathbb{R}^d).$$

We start by showing that $\mathcal{B}_{\mathcal{W}}^*(\mathbb{R}^d)$ and $\mathcal{B}_{\mathcal{V}}^*(\mathbb{R}^d)$ are Schwartz if we assume that \mathcal{W} and \mathcal{V} satisfy (9.1.2) and (9.1.1), respectively. For this, we need the following lemma.

Lemma 11.2.1. *Let M_p be a weight sequence, let v and w be positive functions on \mathbb{R}^d such that v/w vanishes at ∞ , and let $0 < h < k$. Then, the inclusion mapping $\mathcal{B}_w^{M_p, h}(\mathbb{R}^d) \rightarrow \mathcal{B}_v^{M_p, k}(\mathbb{R}^d)$ is compact.*

Proof. This is a consequence of the Arzela-Ascoli theorem (cf. [89, Prop. 2.2]). \square

Proposition 11.2.2. *Let M_p be a weight sequence and let \mathcal{W} be an increasing weight system satisfying (9.1.2). Then, $\mathcal{B}_{\mathcal{W}}^{(M_p)}(\mathbb{R}^d)$ is an (FS) -space while $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$ is an (LFS) -space.*

Proof. Lemma 11.2.1 directly implies that $\mathcal{B}_{\mathcal{W}}^{(M_p)}(\mathbb{R}^d)$ is an (FS) -space. Next, we consider $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$. We may assume that w_n/w_{n+1} vanishes at ∞ for all $n \in \mathbb{N}$. The following isomorphism of l.c.s. holds

$$\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d) \cong \varinjlim_{N \in \mathbb{N}} \varprojlim_{n \in \mathbb{N}} \mathcal{B}_{w_n}^{N+1/n}(\mathbb{R}^d).$$

Furthermore, the Fréchet spaces $\varprojlim_n \mathcal{B}_{w_n}^{N+1/n}(\mathbb{R}^d)$ are Schwartz because of Lemma 11.2.1. \square

Proposition 11.2.3. *Let M_p be a weight sequence and let \mathcal{V} be a decreasing weight system satisfying (9.1.1). Then, $\mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d)$ is an (LFS) -space while $\mathcal{B}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d)$ is a (DFS) -space.*

Proof. Lemma 11.2.1 directly implies that $\mathcal{B}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d)$ is a (DFS) -space. Next, we consider $\mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d)$. We may assume that v_{N+1}/v_N vanishes at ∞ for all $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$ and set $v_N^1 := \sqrt{v_N v_{N+1}}$. Notice that v_N^1 is a positive function such that $v_{N+1} \leq v_N^1 \leq v_N$ and v_{N+1}/v_N^1 vanishes at ∞ . Hence, we can inductively define a sequence $(v_N^n)_n$ of positive functions such that $v_{N+1} \leq v_N^1 \leq \dots \leq v_N^n \leq v_N^{n+1} \leq \dots \leq v_N$ and v_N^n/v_N^{n+1} vanishes at ∞ for each $n \in \mathbb{N}$. We have the following isomorphism of l.c.s.

$$\mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d) \cong \varinjlim_{N \in \mathbb{N}} \varprojlim_{n \in \mathbb{N}} \mathcal{B}_{v_N^n}^{M_p, 1/n}(\mathbb{R}^d).$$

Furthermore, the Fréchet spaces $\varprojlim_n \mathcal{B}_{v_N^n}^{M_p, 1/n}(\mathbb{R}^d)$ are Schwartz because of Lemma 11.2.1. \square

Throughout this chapter the spaces $\mathcal{S}_{\dagger}^*(\mathbb{R}^d)$ and $\mathcal{S}'_{\dagger}(\mathbb{R}^d)$ will play the same role as $\mathcal{D}(\mathbb{R}^d)$ and $\mathcal{D}'(\mathbb{R}^d)$ did in Chapter 10. We

now introduce a set of conditions on \mathcal{W} and \mathcal{V} that are the natural analogues of (10.2.1) and (10.3.1) in the present setting. Let A_p be a weight sequence. An increasing weight system $\mathcal{W} = (w_n)_n$ is said to be (A_p) -*admissible* if

$$\begin{aligned} \forall n \exists \lambda > 0 \exists \tilde{n} \geq n \exists C > 0 \forall x, y \in \mathbb{R}^d : \\ w_n(x + y) \leq C w_{\tilde{n}}(x) e^{A(y/\lambda)} \end{aligned}$$

while it is said to be $\{A_p\}$ -*admissible* if

$$\begin{aligned} \forall n \forall \lambda > 0 \exists \tilde{n} \geq n \exists C > 0 \forall x, y \in \mathbb{R}^d : \\ w_n(x + y) \leq C w_{\tilde{n}}(x) e^{A(y/\lambda)}. \end{aligned}$$

Likewise, a decreasing weight system $\mathcal{V} = (v_N)_N$ is said to be (A_p) -*admissible* if

$$\begin{aligned} \forall N \exists \lambda > 0 \exists \tilde{N} \geq N \exists C > 0 \forall x, y \in \mathbb{R}^d : \\ v_{\tilde{N}}(x + y) \leq C v_N(x) e^{A(y/\lambda)} \end{aligned}$$

while it is said to be $\{A_p\}$ -*admissible* if

$$\begin{aligned} \forall N \forall \lambda > 0 \exists \tilde{N} \geq N \exists C > 0 \forall x, y \in \mathbb{R}^d : \\ v_{\tilde{N}}(x + y) \leq C v_N(x) e^{A(y/\lambda)}. \end{aligned}$$

A first important consequence of these conditions is given in the next two results.

Proposition 11.2.4. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and suppose that $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \neq \{0\}$. Let \mathcal{W} be a \dagger -admissible increasing weight system satisfying (9.1.2). Then, we have the dense continuous inclusion $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \hookrightarrow \mathcal{B}_{\mathcal{W}}(\mathbb{R}^d)$.*

Proposition 11.2.5. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and suppose that $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \neq \{0\}$. Let \mathcal{V} be a \dagger -admissible decreasing weight system satisfying (9.1.1). Then, we have the dense continuous inclusion $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \hookrightarrow \mathcal{B}_{\mathcal{V}}^*(\mathbb{R}^d)$.*

By Propositions 11.2.4 and 11.2.5 we may view $(\mathcal{B}_{\mathcal{W}}^*(\mathbb{R}^d))'$ and $(\mathcal{B}_{\mathcal{V}}^*(\mathbb{R}^d))'$ as subspaces of $\mathcal{S}'_+(\mathbb{R}^d)$. Both these propositions follow from the ensuing lemma.

Lemma 11.2.6. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and suppose that $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \neq \{0\}$. Let v and w be positive functions on \mathbb{R}^d such that v/w vanishes at ∞ and*

$$v(x+y) \leq Cw(x)e^{A(y/\lambda)}, \quad x, y \in \mathbb{R}^d, \quad (11.2.1)$$

for some $C, \lambda > 0$. Then, for $0 < h < k/H$, the space $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ is dense in $\mathcal{B}_w^{M_p, h}(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{\mathcal{B}_v^{M_p, k}}$.

Proof. Let $\varphi \in \mathcal{B}_w^{M_p, h}(\mathbb{R}^d)$ be arbitrary. Choose $\psi, \chi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ with $\psi(0) = 1$ and $\int_{\mathbb{R}^d} \chi(x) dx = 1$. We define $\psi_n = \psi(\cdot/n)$ and $\chi_n = n^d \chi(n\cdot)$, $n \geq 1$. Set $\varphi_n = \chi_n * (\psi_n \varphi) \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. We claim that $\varphi_n \rightarrow \varphi$ in $\mathcal{B}_v^{M_p, k}(\mathbb{R}^d)$. Choose $l > 0$ so small that $h+l \leq k/H$. Notice that

$$\|\varphi_n - \varphi\|_{\mathcal{B}_v^{M_p, k}} \leq \|\varphi_n - \psi_n \varphi\|_{\mathcal{B}_v^{M_p, k}} + \|\psi_n \varphi - \varphi\|_{\mathcal{B}_v^{M_p, k}}. \quad (11.2.2)$$

We start by estimating the second term in the right-hand side of (11.2.2). It holds that

$$\begin{aligned} & \|\psi_n \varphi - \varphi\|_{\mathcal{B}_v^{M_p, k}} \\ & \leq \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{v(x)}{k^{|\alpha|} M_\alpha} |\psi(x/n) - 1| |\partial^\alpha \varphi(x)| \\ & \leq \|\varphi\|_{\mathcal{B}_w^{M_p, k}} \sup_{x \in \mathbb{R}^d} \frac{v(x)}{w(x)} |\psi(x/n) - 1| + \frac{1}{n} \|\psi\|_{\mathcal{S}_{A_p, 0}^{M_p, l}} \|\varphi\|_{\mathcal{B}_v^{M_p, h}}, \end{aligned}$$

which tends to zero because $\psi(0) = 1$ and v/w vanishes at ∞ . Next, we estimate the first term in the right-hand side of (11.2.2). Observe that

$$\begin{aligned} \|\psi_n \varphi\|_{\mathcal{B}_w^{M_p, k/H}} & \leq \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{H^{|\alpha|} w(x)}{k^{|\alpha|} M_\alpha} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^\beta \psi(x/n)| |\partial^{\alpha-\beta} \varphi(x)| \\ & \leq \|\psi\|_{\mathcal{S}_{A_p, 0}^{M_p, l}} \|\varphi\|_{\mathcal{B}_w^{M_p, h}} \end{aligned}$$

for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
& \|\varphi_n - \psi_n \varphi\|_{\mathcal{B}_v^{M_p, k}} \\
& \leq \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{v(x)}{k^{|\alpha|} M_\alpha} \int_{\mathbb{R}^d} |\chi(t)| |\partial^\alpha(\psi_n \varphi)(x - (t/n)) - \partial^\alpha(\psi_n \varphi)(x)| dt \\
& \leq \frac{1}{n} \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{v(x)}{k^{|\alpha|} M_\alpha} \int_{\mathbb{R}^d} |\chi(t)| |t| \cdot \\
& \quad \sum_{|\beta|=1} \int_0^1 |\partial^{\alpha+\beta}(\psi_n \varphi)(x - (\gamma t/n))| d\gamma dt \\
& \leq \frac{1}{n} dk C_0 C M_1 \|\psi_n \varphi\|_{\mathcal{B}_w^{M_p, k/H}} \int_{\mathbb{R}^d} |\chi(t)| |t| e^{A(t/\lambda)} dt \\
& \leq \frac{1}{n} dk C_0 C M_1 \|\psi\|_{\mathcal{S}_{A_p, 0}^{M_p, l}} \|\varphi\|_{\mathcal{B}_w^{M_p, h}} \int_{\mathbb{R}^d} |\chi(t)| |t| e^{A(t/\lambda)} dt. \quad \square
\end{aligned}$$

11.2.2 Regularity properties

We now discuss the regularity properties of the spaces $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$ and $\mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d)$.

Theorem 11.2.7. *Let M_p be a weight sequence and let $\mathcal{W} = (w_n)_n$ be an increasing weight system satisfying (10.3.1). Consider the following conditions:*

- (i) \mathcal{W} satisfies (DN) .
- (ii) $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$ is boundedly retractive.
- (iii) $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$ satisfies (wQ) .

Then, (i) \Rightarrow (ii) \Leftrightarrow (iii). Moreover, if M_p satisfies (M.1), (M.2), and (M.3), then (iii) \Rightarrow (i).

Theorem 11.2.8. *Let M_p be a weight sequence and let $\mathcal{V} = (v_N)_N$ be a decreasing weight system satisfying (10.2.1). Consider the following conditions:*

(i) \mathcal{V} satisfies (Ω) .

(ii) $\mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d)$ is boundedly retractive.

(iii) $\mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d)$ satisfies (wQ) .

Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$ and, if \mathcal{V} is regularly decreasing, $(iii) \Rightarrow (ii)$. Moreover, if M_p satisfies (M.1), (M.2), and (M.3), then $(iii) \Rightarrow (i)$.

The proofs of Theorems 11.2.7 and 11.2.8 are based on Theorem 2.3.1 and Lemmas 9.1.15 and 9.1.13, respectively. The main new idea is to use the result about weighted inductive limits of spaces of vector-valued continuous functions presented in Subsection 9.1.2 to show that $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$ and $\mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d)$ are boundedly retractive. We need two lemmas in preparation.

Lemma 11.2.9. *Let M_p be a weight sequence and let $\mathcal{W} = (w_n)_n$ be an increasing weight system. Then, $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$ is boundedly stable.*

Proof. Let $h > 0$ be arbitrary and let B be a bounded set of $\mathcal{B}_{\mathcal{W}}^{M_p, h}(\mathbb{R}^d)$. We shall show that, for all $0 < h < k \leq l$, the spaces $\mathcal{B}_{\mathcal{W}}^{M_p, k}(\mathbb{R}^d)$ and $\mathcal{B}_{\mathcal{W}}^{M_p, l}(\mathbb{R}^d)$ induce the same topology on B . We only need to prove that the topology induced by $\mathcal{B}_{\mathcal{W}}^{M_p, l}(\mathbb{R}^d)$ is finer than the one induced by $\mathcal{B}_{\mathcal{W}}^{M_p, k}(\mathbb{R}^d)$. Consider the basis of neighbourhoods of 0 in $\mathcal{B}_{\mathcal{W}}^{M_p, k}(\mathbb{R}^d)$ given by

$$U(n, \varepsilon) = \{\varphi \in \mathcal{B}_{\mathcal{W}}^{M_p, k}(\mathbb{R}^d) \mid \|\varphi\|_{\mathcal{B}_{w_n}^{M_p, k}} \leq \varepsilon\}, \quad n \in \mathbb{N}, \varepsilon > 0.$$

Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. Choose $n_0 \in \mathbb{N}$ so large that $(h/k)^{n_0} \leq \varepsilon(\sup_{\varphi \in B} \|\varphi\|_{\mathcal{B}_{w_n}^{M_p, h}})^{-1}$. Set $\delta = (k/l)^{n_0} \varepsilon$ and $V = \{\varphi \in \mathcal{B}_{\mathcal{W}}^{M_p, l}(\mathbb{R}^d) \mid \|\varphi\|_{\mathcal{B}_{w_n}^{M_p, l}} \leq \delta\}$. We claim that $B \cap V \subseteq U(n, \varepsilon)$. Indeed, for $\varphi \in B \cap V$ it holds that

$$\|\varphi\|_{\mathcal{B}_{w_n}^{M_p, k}} = \max \left\{ \sup_{|\alpha| \leq n_0} \frac{\|\partial^\alpha \varphi\|_{Cw_n}}{k^{|\alpha|} M_\alpha}, \sup_{|\alpha| \geq n_0} \frac{\|\partial^\alpha \varphi\|_{Cw_n}}{k^{|\alpha|} M_\alpha} \right\} \leq \varepsilon.$$

□

Lemma 11.2.10. *Let M_p be a weight sequence and let $\mathcal{V} = (v_N)_N$ be a regularly decreasing weight system. Then, $\mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d)$ is boundedly stable.*

Proof. The (LB) -space $\mathcal{VC}(\mathbb{R}^d)$ is boundedly stable because \mathcal{V} is regularly decreasing (Proposition 9.1.1). Let $N \in \mathbb{N}$ be arbitrary and choose $M \geq N$ such that, for all $K \geq M$, the spaces $Cv_M(\mathbb{R}^d)$ and $Cv_K(\mathbb{R}^d)$ induce the same topology on the bounded sets of $Cv_N(\mathbb{R}^d)$. We shall show that, for all $K \geq M$, the spaces $\mathcal{B}_{v_M}^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{B}_{v_K}^{(M_p)}(\mathbb{R}^d)$ induce the same topology on the bounded sets B of $\mathcal{B}_{v_N}^{(M_p)}(\mathbb{R}^d)$. We only need to prove that the topology induced by $\mathcal{B}_{v_K}^{(M_p)}(\mathbb{R}^d)$ is finer than the one induced by $\mathcal{B}_{v_M}^{(M_p)}(\mathbb{R}^d)$. Consider the basis of neighbourhoods of 0 in $\mathcal{B}_{v_M}^{(M_p)}(\mathbb{R}^d)$ given by

$$U(h, \varepsilon) = \{\varphi \in \mathcal{B}_{v_M}^{(M_p)}(\mathbb{R}^d) \mid \|\varphi\|_{\mathcal{B}_{v_M}^{M_p, h}} \leq \varepsilon\}, \quad h, \varepsilon > 0.$$

Let $h, \varepsilon > 0$ be arbitrary. Since B is a bounded subset of $\mathcal{B}_{v_N}^{(M_p)}(\mathbb{R}^d)$, the set

$$B' = \left\{ \frac{\partial^\alpha \varphi}{h^{|\alpha|} M_\alpha} \mid \alpha \in \mathbb{N}^d, \varphi \in B \right\}$$

is bounded in $Cv_N(\mathbb{R}^d)$. Set $U' = \{f \in Cv_M(\mathbb{R}^d) \mid \|f\|_{Cv_M} \leq \varepsilon\}$. As $Cv_M(\mathbb{R}^d)$ and $Cv_K(\mathbb{R}^d)$ induce the same topology on B' , there is $\delta > 0$ such that $B' \cap V' \subseteq U'$, where $V' = \{f \in Cv_K(\mathbb{R}^d) \mid \|f\|_{Cv_K} \leq \delta\}$. Finally, set $V = \{\varphi \in \mathcal{B}_{v_K}^{(M_p)}(\mathbb{R}^d) \mid \|\varphi\|_{\mathcal{B}_{v_K}^{M_p, h}} \leq \delta\}$ and notice that $B \cap V \subseteq U$. \square

Proof of Theorem 11.2.7. The implication $(ii) \Rightarrow (iii)$ holds for general (LF) -spaces while $(iii) \Rightarrow (ii)$ follows from Theorem 2.3.1 and Lemma 11.2.9. We now show $(i) \Rightarrow (ii)$. Set $E = \mathcal{WC}(\mathbb{R}^d)$. We first assume that E is normable. Then, there is $n_0 \in \mathbb{N}$ such that E is topologically isomorphic to $Cw_{n_0}(\mathbb{R}^d)$. Consequently, $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$ is topologically isomorphic to the (LB) -space $\mathcal{B}_{w_{n_0}}^{\{M_p\}}(\mathbb{R}^d)$. By applying Proposition 11.2.9 to the constant weight system $\mathcal{W} = (w_n)_n$, we obtain that $\mathcal{B}_{w_{n_0}}^{\{M_p\}}(\mathbb{R}^d)$ is boundedly stable. Since every (LB) -space satisfies (wQ) , Theorem 2.3.1 yields that $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$ is boundedly retractive. Next, we assume that E is non-normable. It suffices

to show that $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$ is sequentially retractive. Let $(\varphi_j)_j$ be a null sequence in $\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$. Consider the decreasing weight system $\mathcal{V} = (v_N)_N$ on \mathbb{N}^d (endowed with the discrete topology) given by $v_N(\alpha) = N^{-|\alpha|}$ and observe that the mapping

$$T : \mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d) \rightarrow \mathcal{VC}(\mathbb{N}^d; E) : \varphi \rightarrow \left(\frac{\partial^\alpha \varphi}{M_\alpha} \right)_\alpha$$

is continuous. Since \mathcal{V} satisfies (Ω) and E satisfies (DN) , Theorem 9.1.9 and Lemma 9.1.12 imply that $\mathcal{VC}(\mathbb{N}^d; E)$ is sequentially retractive. Therefore, the sequence $(T(\varphi_j))_j$ is contained and converges to zero in $Cv_N(\mathbb{N}^d; E)$ for some $N \in \mathbb{N}$, whence $(\varphi_j)_j$ is contained and converges to zero in $\mathcal{B}_{\mathcal{W}}^{M_p, N}(\mathbb{R}^d)$. Finally, we assume that M_p satisfies $(M.1)$, $(M.2)$, and $(M.3)$ and show $(iii) \Rightarrow (i)$. By [113, Cor. 4.10] we have that $\mathcal{D}_{[-1,1]^d}^{\{M_p\}} \cong \Lambda'_0(\beta)$, where $\beta = (M(j^{1/d}))_j$. The sequence β is shift-stable because of [89, Lemma 4.1]. Hence, by Remark 9.1.14 and Lemma 9.1.15, it suffices to show that $(\mathcal{W}, \mathcal{D}_{[-1,1]^d}^{\{M_p\}})$ satisfies $(S_2)^*$. This can be done as in the last part of the proof of Theorem 10.2.1. \square

Proof of Theorem 11.2.8. Again, the implication $(ii) \Rightarrow (iii)$ holds for general (LF) -spaces while $(iii) \Rightarrow (ii)$ (under the extra assumption that \mathcal{V} is regularly decreasing) follows from Theorem 2.3.1 and Lemma 11.2.10. Next, we show $(i) \Rightarrow (ii)$. It suffices to show that $\mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d)$ is sequentially retractive. Let $(\varphi_j)_j$ be a null sequence in $\mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d)$. Define E as the Fréchet space consisting of all multi-indexed sequences $(c_\alpha)_\alpha \in \mathbb{C}^{\mathbb{N}^d}$ such that $\sup_\alpha |c_\alpha|/h^{|\alpha|} < \infty$ for all $h > 0$ and observe that the mapping

$$T : \mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d) \rightarrow \mathcal{VC}(\mathbb{R}^d; E) : \varphi \mapsto \left(\frac{\partial^\alpha \varphi}{M_\alpha} \right)_\alpha$$

is continuous. Since \mathcal{V} satisfies (Ω) and E satisfies (DN) , Theorem 9.1.9 and Lemma 9.1.12 imply that $\mathcal{VC}(\mathbb{N}^d; E)$ is sequentially retractive. Therefore, the sequence $(T(\varphi_j))_j$ is contained and converges to zero in $Cv_N(\mathbb{N}^d; E)$ for some $N \in \mathbb{N}$, whence

$(\varphi_j)_j$ is contained and converges to zero in $\mathcal{B}_{v_N}^{(M_p)}(\mathbb{R}^d)$. Finally, we assume that M_p satisfies (M.1), (M.2), and (M.3) and show (iii) \Rightarrow (i). By [113, Cor. 4.3] we have that $\mathcal{D}_{[-1,1]^d}^{(M_p)} \cong \Lambda_\infty(\beta)$, where $\beta = (M(j^{1/d}))_j$. The sequence β is shift-stable because of [89, Lemma 4.1]. Hence, by Remark 9.1.10 and Lemma 9.1.13, it suffices to show that $(\mathcal{D}_{[-1,1]^d}^{(M_p)}, \mathcal{V})$ satisfies $(S_2)^*$. This can be done as in the last part of the proof of Theorem 10.2.1. \square

11.2.3 Characterization via the STFT

We now turn our attention to the mapping properties of the STFT on the spaces $\mathcal{B}_{\mathcal{W}}^*(\mathbb{R}^d)$ and $\mathcal{B}_{\mathcal{V}}^*(\mathbb{R}^d)$. We recall that (M.2)' implies that (cf. Subsection 2.2.1)

$$M(H^\lambda t) - M(t) \geq \lambda \log(t/C_0), \quad t \geq 0,$$

for all $\lambda > 0$. In particular,

$$e^{M(t)-M(H^{d+1}t)} \leq \frac{(2C_0)^{d+1}}{(1+t)^{d+1}}, \quad t \geq 0.$$

Lemma 11.2.11. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. Let v and w be non-negative measurable functions on \mathbb{R}^d satisfying (11.2.1). Then,*

$$V_\psi : \mathcal{B}_w^{M_p, h}(\mathbb{R}^d) \rightarrow C v \otimes e^{M(\pi \cdot / (\sqrt{d}h))}(\mathbb{R}_{x, \xi}^{2d})$$

is a well-defined continuous mapping.

Proof. Let $\varphi \in \mathcal{B}_w^{M_p, h}(\mathbb{R}^d)$ be arbitrary. Then,

$$\begin{aligned} & |\xi^\alpha V_\psi \varphi(x, \xi)| v(x) \\ & \leq C(2\pi)^{-|\alpha|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} |\partial^\beta \varphi(t)| w(t) |\partial^{\alpha-\beta} \psi(t-x)| e^{A((t-x)/\lambda)} dt \\ & \leq C' \|\varphi\|_{\mathcal{B}_w^{M_p, h}} (h/\pi)^{|\alpha|} M_\alpha \end{aligned}$$

for all $(x, \xi) \in \mathbb{R}^{2d}$ and $\alpha \in \mathbb{N}^d$, where

$$C' = (2C_0)^{d+1} C \|\psi\|_{\mathcal{S}_{A_p, \lambda/H^{d+1}}^{M_p, h}} \int_{\mathbb{R}^d} \frac{1}{(1 + |t|/\lambda)^{d+1}} dt < \infty.$$

Hence,

$$\begin{aligned} |V_\psi \varphi(x, \xi)| v(x) &\leq C' \|\varphi\|_{\mathcal{B}_w^{M_p, h}} \inf_{p \in \mathbb{N}} \frac{M_p}{(\pi |\xi| / (\sqrt{d}h))^p} \\ &= C' M_0 \|\varphi\|_{\mathcal{B}_w^{M_p, h}} e^{-M(\pi \xi / (\sqrt{d}h))}. \end{aligned}$$

□

Lemma 11.2.12. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. Let v and w be non-negative measurable functions on \mathbb{R}^d satisfying (11.2.1). Then,*

$$V_\psi^* : Cw \otimes e^{M(\cdot/h)}(\mathbb{R}_{x, \xi}^{2d}) \rightarrow \mathcal{B}_v^{M_p, 4\pi H^{d+1}h}(\mathbb{R}^d)$$

is a well-defined continuous mapping.

Proof. For simplicity we write $\|\cdot\|_{Cw \otimes e^{M(\cdot/h)}} = \|\cdot\|$. Let $F \in Cw \otimes e^{M(\cdot/h)}(\mathbb{R}_{x, \xi}^{2d})$ be arbitrary. Then,

$$\begin{aligned} &\sup_{t \in \mathbb{R}^d} |\partial^\alpha V_\psi^* F(t)| v(t) \\ &\leq C \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sup_{t \in \mathbb{R}^d} \int \int_{\mathbb{R}^{2d}} |F(x, \xi)| w(x) (2\pi |\xi|)^{|\beta|} \\ &\quad |\partial^{\alpha-\beta} \psi(t-x)| e^{A((t-x)/\lambda)} dx d\xi \\ &\leq C' \|F\| M_\alpha (4\pi H^{d+1}h)^{|\alpha|} \end{aligned}$$

for all $\alpha \in \mathbb{N}^d$, where the constant C' is given by

$$\begin{aligned} &\frac{(2C_0)^{2(d+1)} C}{M_0} \|\psi\|_{\mathcal{S}_{A_p, \lambda/H^{d+1}}^{M_p, 2\pi H^{d+1}h}} \\ &\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|/(H^{d+1}h))^{d+1}} d\xi \int_{\mathbb{R}^d} \frac{1}{(1 + |t|/\lambda)^{d+1}} dt < \infty. \end{aligned}$$

□

We define $\mathcal{W}_{(M_p)} := (e^{M(n \cdot)})_n$ and $\mathcal{V}_{\{M_p\}} := (e^{M(\cdot/N)})_N$, an increasing and decreasing weight system on \mathbb{R}^d , respectively. Lemmas 11.2.11 and 11.2.12 directly imply the following result.

Proposition 11.2.13. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. Let $\mathcal{W} = (w_n)_n$ be an (A_p) -admissible increasing weight system.*

(i) *The mappings*

$$V_\psi : \mathcal{B}_{\mathcal{W}}^{(M_p)}(\mathbb{R}^d) \rightarrow \mathcal{W} \otimes \mathcal{W}_{(M_p)} C(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi^* : \mathcal{W} \otimes \mathcal{W}_{(M_p)} C(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \mathcal{B}_{\mathcal{W}}^{(M_p)}(\mathbb{R}^d)$$

are well-defined and continuous.

(ii) *The mappings*

$$V_\psi : \mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d) \rightarrow \mathcal{W} \otimes_i \mathcal{V}_{\{M_p\}} C(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi^* : \mathcal{W} \otimes_i \mathcal{V}_{\{M_p\}} C(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d)$$

are well-defined and continuous.

Proposition 11.2.14. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. Let $\mathcal{V} = (v_N)_N$ be an (A_p) -admissible decreasing weight system.*

(i) *The mappings*

$$V_\psi : \mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d) \rightarrow \mathcal{V} \otimes_i \mathcal{W}_{(M_p)} C(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi^* : \mathcal{V} \otimes_i \mathcal{W}_{(M_p)} C(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \mathcal{B}_{\mathcal{V}}^{(M_p)}(\mathbb{R}^d)$$

are well-defined and continuous.

(ii) *The mappings*

$$V_\psi : \mathcal{B}_\mathcal{V}^{\{M_p\}}(\mathbb{R}^d) \rightarrow \mathcal{V} \otimes \mathcal{V}_{\{M_p\}} C(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi^* : \mathcal{V} \otimes \mathcal{V}_{\{M_p\}} C(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \mathcal{B}_\mathcal{V}^{\{M_p\}}(\mathbb{R}^d)$$

are well-defined and continuous.

11.2.4 A projective description

Our next goal is to give a projective description of the (LB) -space $\mathcal{B}_\mathcal{V}^{\{M_p\}}(\mathbb{R}^d)$ in terms of Komatsu's family \mathfrak{R} and the maximal Nachbin family associated to \mathcal{V} . Throughout this subsection we shall work with decreasing weight systems $\mathcal{V} = (v_N)_N$ satisfying the ensuing condition

$$\begin{aligned} \exists \lambda > 0 \forall N \exists \tilde{N} \geq N \exists C > 0 \forall x, y \in \mathbb{R}^d : \\ v_{\tilde{N}}(x + y) \leq C v_N(x) e^{A(y/\lambda)}. \end{aligned} \quad (11.2.3)$$

Notice that (11.2.4) is stronger than (A_p) -admissibility but weaker than $\{A_p\}$ -admissibility.

Theorem 11.2.15. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and suppose that $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \neq \{0\}$. Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system satisfying (11.2.4) and condition (V) (cf. Theorem 9.1.3). Then, $\varphi \in C^\infty(\mathbb{R}^d)$ belongs to $\mathcal{B}_\mathcal{V}^{\{M_p\}}(\mathbb{R}^d)$ if and only if*

$$\|\varphi\|_{\mathcal{B}_v^{M_p, h_j}} := \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)| v(x)}{M_\alpha \prod_{j=0}^{|\alpha|} h_j} < \infty$$

for all $h_j \in \mathfrak{R}$ and $v \in \overline{V}$. Moreover, the topology of $\mathcal{B}_\mathcal{V}^{\{M_p\}}(\mathbb{R}^d)$ is generated by the system of seminorms $\{\|\cdot\|_{\mathcal{B}_v^{M_p, h_j}} \mid h_j \in \mathfrak{R}, v \in \overline{V}\}$.

We write $\tilde{\mathcal{B}}_\mathcal{V}^{\{M_p\}}(\mathbb{R}^d)$ for the space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\|\varphi\|_{\mathcal{B}_v^{M_p, h_j}} < \infty$ for all $h_j \in \mathfrak{R}$ and $v \in \overline{V}$ and endow it

with the locally convex topology generated by the system of seminorms $\{\|\cdot\|_{\mathcal{B}_v^{M_p, h_j}} \mid h_j \in \mathfrak{R}, v \in \overline{V}\}$. Of course, Theorem 11.2.15 then asserts that $\mathcal{B}_v^{\{M_p\}}(\mathbb{R}^d) = \widetilde{\mathcal{B}}_v^{\{M_p\}}(\mathbb{R}^d)$ topologically. The next lemma shows that these spaces always coincide algebraically.

Lemma 11.2.16. *Let M_p be a weight sequence and let \mathcal{V} be a decreasing weight system. Then, $\mathcal{B}_v^{\{M_p\}}(\mathbb{R}^d)$ and $\widetilde{\mathcal{B}}_v^{\{M_p\}}(\mathbb{R}^d)$ coincide algebraically and the inclusion mapping $\mathcal{B}_v^{\{M_p\}}(\mathbb{R}^d) \rightarrow \widetilde{\mathcal{B}}_v^{\{M_p\}}(\mathbb{R}^d)$ is continuous.*

Proof. It is obvious that $\mathcal{B}_v^{\{M_p\}}(\mathbb{R}^d)$ is continuously included in $\widetilde{\mathcal{B}}_v^{\{M_p\}}(\mathbb{R}^d)$. For the converse inclusion we consider the decreasing weight system $\mathcal{U} = (u_N)_N$ on \mathbb{N}^d (endowed with the discrete topology) given by $u_N(\alpha) := N^{-|\alpha|}$. Now let $\varphi \in \widetilde{\mathcal{B}}_v^{\{M_p\}}(\mathbb{R}^d)$ be arbitrary and define $f(x, \alpha) = \partial^\alpha \varphi(x) / M_\alpha$ for $x \in \mathbb{R}^d$, $\alpha \in \mathbb{N}^d$. By Lemma 9.1.6 and Lemma 2.2.2(i) we have that $f \in C\overline{V}(\mathcal{V} \otimes \mathcal{U})(\mathbb{R}^d \times \mathbb{N}^d)$. Since $\mathcal{V} \otimes \mathcal{UC}(\mathbb{R}^d \times \mathbb{N}^d) = C\overline{V}(\mathcal{V} \otimes \mathcal{U})(\mathbb{R}^d \times \mathbb{N}^d)$ as sets (cf. Subsection 9.1.1), we obtain that $f \in \mathcal{V} \otimes \mathcal{UC}(\mathbb{R}^d \times \mathbb{N}^d)$ and, thus, $\varphi \in \mathcal{B}_v^{\{M_p\}}(\mathbb{R}^d)$. \square

Next, we discuss the mapping properties of the STFT on the space $\widetilde{\mathcal{B}}_v^{\{M_p\}}(\mathbb{R}^d)$. The following technical lemma is needed.

Lemma 11.2.17. *Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system satisfying (11.2.4). Then, for every $v \in \overline{V}$ there is $\bar{v} \in \overline{V}$ such that $v(x + y) \leq \bar{v}(x)e^{A(y/\lambda)}$ for all $x, y \in \mathbb{R}^d$.*

Proof. Find a strictly increasing sequence of natural numbers $(N_j)_j$ such that $v_{N_{j+1}}(x + y) \leq C_j v_{N_j}(x) e^{A(y/\lambda)}$ for all $x, y \in \mathbb{R}^d$ and some $C_j > 0$. Pick $C'_j > 0$ such that $v \leq C'_j v_{N_j}$ for all $j \in \mathbb{N}$. Set $\bar{v} = \inf_j C_j C'_{j+1} v_{N_j} \in \overline{V}$. We have that

$$\begin{aligned} v(x + y) &\leq \inf_{j \in \mathbb{N}} C'_{j+1} v_{N_{j+1}}(x + y) \\ &\leq e^{A(y/\lambda)} \inf_{j \in \mathbb{N}} C_j C'_{j+1} v_{N_j}(x) = \bar{v}(x) e^{A(y/\lambda)}. \end{aligned}$$

\square

Proposition 11.2.18. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. Let $\mathcal{V} = (v_N)_N$ be a decreasing weight system satisfying (11.2.4). Then,*

$$V_\psi : \tilde{\mathcal{B}}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d) \rightarrow C\bar{V}(\mathcal{V} \otimes \mathcal{V}_{\{M_p\}})(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi^* : C\bar{V}(\mathcal{V} \otimes \mathcal{V}_{\{M_p\}})(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \tilde{\mathcal{B}}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d)$$

are well-defined continuous mappings.

Proof. Let $u \in \bar{V}(\mathcal{V} \otimes \mathcal{V}_{\{M_p\}})$ be arbitrary. Lemmas 9.1.6 and 2.2.4 imply that there are $v \in \bar{V}(\mathcal{V})$ and $h_j \in \mathfrak{R}$ such that $u \leq v \otimes e^{M_{h_j}}$. By Lemma 2.2.3 we may assume that the weight sequence $M_p \prod_{j=0}^p h_j$ satisfies (M.1) and (M.2)'. Next, by Lemma 11.2.17 there is $\bar{v} \in \bar{V}$ such that $v(x+y) \leq \bar{v}(x)e^{A(y/\lambda)}$ for all $x, y \in \mathbb{R}^d$. Set $h'_j = h_j \pi / \sqrt{d}$. Lemma 11.2.11 implies that the mapping $V_\psi : \mathcal{B}_{\bar{v}}^{M_p, h'_j}(\mathbb{R}^d) \rightarrow Cv \otimes e^{M_{h_j}}(\mathbb{R}_{x,\xi}^{2d})$ is well-defined and continuous. As the inclusion mapping $Cv \otimes e^{M_{h_j}}(\mathbb{R}_{x,\xi}^{2d}) \rightarrow Cu(\mathbb{R}_{x,\xi}^{2d})$ is continuous, we may conclude that V_ψ is well-defined and continuous. The assertion concerning V_ψ^* follows similarly from Lemma 11.2.12. \square

Proof of Theorem 11.2.15. We need to show that the topological equality $\mathcal{B}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d) = \tilde{\mathcal{B}}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d)$. By Lemma 11.2.16 it suffices to show that the inclusion mapping $\iota : \tilde{\mathcal{B}}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d) \rightarrow \mathcal{B}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d)$ is continuous. Since M_p satisfies (M.2)', the decreasing weight system $\mathcal{V}_{\{M_p\}}$ satisfies (S) and, thus, condition (V) (Remark 9.1.5). Hence, Theorem 9.1.3 and Lemma 9.1.7 yield that $\mathcal{V} \otimes \mathcal{V}_{\{M_p\}}C(\mathbb{R}_{x,\xi}^{2d}) = C\bar{V}(\mathcal{V} \otimes \mathcal{V}_{\{M_p\}})(\mathbb{R}_{x,\xi}^{2d})$ topologically. Choose $\psi, \gamma \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ such that $(\gamma, \psi)_{L^2} = 1$. By (9.3.4) the following diagram commutes

$$\begin{array}{ccc} \tilde{\mathcal{B}}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d) & \xrightarrow{V_\psi} & C\bar{V}(\mathcal{V} \otimes \mathcal{V}_{\{M_p\}})(\mathbb{R}_{x,\xi}^{2d}) = \mathcal{V} \otimes \mathcal{V}_{\{M_p\}}C(\mathbb{R}_{x,\xi}^{2d}) \\ \downarrow \iota & \swarrow V_\gamma^* & \\ \mathcal{B}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d) & & \end{array}$$

Propositions 11.2.14 and 11.2.18 imply that V_ψ and V_γ^* are continuous, whence ι is also continuous. \square

We end this subsection by stating two important particular cases of our main result. Firstly, by applying Theorem 11.2.15 to $\mathcal{V} = \mathcal{V}_{\{A_p\}}$ and using Lemma 2.2.4, we obtain the well-known projective description of the Gelfand-Shilov space $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$.

Proposition 11.2.19. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and suppose that $\mathcal{S}_{\{A_p\}}^{(M_p)}(\mathbb{R}^d) \neq \{0\}$. Then, $\varphi \in C^\infty(\mathbb{R}^d)$ belongs to $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ if and only if*

$$\|\varphi\|_{\mathcal{S}_{A_p, h_j}^{M_p, h_j}} := \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)| e^{A_{h_j}(x)}}{M_\alpha \prod_{j=0}^{|\alpha|} h_j} < \infty$$

for all $h_j \in \mathfrak{R}$. Moreover, the topology of $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ is generated by the system of seminorms $\{\|\cdot\|_{\mathcal{S}_{A_p, h_j}^{M_p, h_j}} \mid h_j \in \mathfrak{R}\}$.

Remark 11.2.20. Proposition 11.2.19 was first shown by Pilipović [132, Lemma 4]. His proof is based on a structural theorem for the dual space $\mathcal{S}'_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ and a dual Mittag-Leffler argument.

Next, we consider weighted spaces of ultradifferentiable functions of Roumieu type.

Theorem 11.2.21. *Let M_p and A_p be weight sequences satisfying (M.1) and (M.2)' and suppose that $\mathcal{S}_{\{A_p\}}^{(M_p)}(\mathbb{R}^d) \neq \{0\}$. Let ω be a positive measurable function on \mathbb{R}^d such that $\omega(x+y) \leq C\omega(x)e^{A(y/\lambda)}$ for all $x, y \in \mathbb{R}^d$ and some $C, \lambda > 0$. Then, $\varphi \in C^\infty(\mathbb{R}^d)$ belongs to $\mathcal{B}_\omega^{\{M_p\}}(\mathbb{R}^d)$ if and only if*

$$\|\varphi\|_{\mathcal{B}_\omega^{M_p, h_j}} = \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)| \omega(x)}{M_\alpha \prod_{j=0}^{|\alpha|} h_j} < \infty$$

for all $h_j \in \mathfrak{R}$. Moreover, the topology of $\mathcal{B}_\omega^{\{M_p\}}(\mathbb{R}^d)$ is generated by the system of seminorms $\{\|\cdot\|_{\mathcal{B}_\omega^{M_p, h_j}} \mid h_j \in \mathfrak{R}\}$.

Proof. We may assume that ω is continuous (cf. the proof of Theorem 10.4.1). Set $\mathcal{V} = (\omega)_N$ and notice that \mathcal{V} satisfies (V) (Remark 9.1.5). The result now follows from Theorem 11.2.15 and the fact that $\overline{V}(\mathcal{V}) = \{\lambda\omega \mid \lambda > 0\}$. \square

Remark 11.2.22. Theorem 11.2.21 was already shown in [53, Thm. 4.17] under much more restrictive conditions on M_p and A_p and with a more complicated proof based on the computation of biduals of weighted $\dot{\mathcal{B}}$ spaces of ultradifferentiable functions of Roumieu type.

11.3 On two classes of weighted convolutor spaces

We introduce two classes of weighted convolutor spaces in this section and study their structural and topological properties. As in Section 10.3, the STFT will be the main tool in our proofs and therefore we start with a discussion about the STFT on these spaces. Throughout this section M_p and A_p will always stand for a pair of weight sequences satisfying (9.2). We also fix a decreasing weight system $\mathcal{V} = (v_n)_n$ satisfying (9.1.1) and

$$\forall n \exists m > n : v_m(x)/v_n(x) = O(g(x)) \quad (11.3.1)$$

for some $g \in L^1(\mathbb{R}^d)$ and an increasing weight system $\mathcal{W} = (w_N)_N$ satisfying (9.1.2) and

$$\forall N \exists M > N : w_N/w_M \in L^1(\mathbb{R}^d). \quad (11.3.2)$$

Furthermore, we assume that \mathcal{V} and \mathcal{W} are (A_p) -admissible in the Beurling case and $\{A_p\}$ -admissible in the Roumieu case.

We are interested in the following convolutor spaces

$$\mathcal{O}'_C(\mathcal{S}_+^*, \mathcal{V}C) := \{f \in \mathcal{S}_+^{*'}(\mathbb{R}^d) \mid f * \varphi \in \mathcal{V}C(\mathbb{R}^d) \text{ for all } \varphi \in \mathcal{S}_+^*(\mathbb{R}^d)\}$$

and

$$\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{WC}) := \{f \in \mathcal{S}_\dagger^{*\prime}(\mathbb{R}^d) \mid f * \varphi \in \mathcal{WC}(\mathbb{R}^d) \text{ for all } \varphi \in \mathcal{S}_\dagger^*(\mathbb{R}^d)\}$$

endowed with the initial topology with respect to the mappings

$$\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{VC}) \rightarrow L_b(\mathcal{S}_\dagger^*(\mathbb{R}^d), \mathcal{VC}(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi)$$

and

$$\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{WC}) \rightarrow L_b(\mathcal{S}_\dagger^*(\mathbb{R}^d), \mathcal{WC}(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi),$$

respectively.

11.3.1 Characterization via the STFT

In this subsection, we discuss the mapping properties of the STFT on the spaces $\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{VC})$ and $\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{WC})$. We start with two lemmas.

Lemma 11.3.1. *Let X and Y be locally compact Hausdorff spaces. Let $\mathcal{V} = (v_n)_n$ be a decreasing weight system on X satisfying (9.1.1) and let $\mathcal{W} = (w_N)_N$ be an increasing weight system on Y . Then, $\mathcal{VC}(X) \widehat{\otimes}_\varepsilon \mathcal{WC}(Y)$ may be identified with the space consisting of all $f \in C(X \times Y)$ satisfying the ensuing property: For all $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that*

$$\sup_{(x,y) \in X \times Y} |f(x,y)| v_n(x) w_N(y) < \infty.$$

Consequently, $f \in C(X \times Y)$ belongs to $\mathcal{VC}(X) \widehat{\otimes}_\varepsilon \mathcal{WC}(Y)$ if and only if

$$\|f\|_{Cv \otimes w_N} = \sup_{(x,y) \in X \times Y} |f(x,y)| v(x) w_N(y) < \infty$$

for all $N \in \mathbb{N}$ and $v \in \overline{V}$. Moreover, the topology of the space $\mathcal{VC}(X) \widehat{\otimes}_\varepsilon \mathcal{WC}(Y)$ is generated by the system of seminorms $\{\|\cdot\|_{Cv \otimes w_N} \mid N \in \mathbb{N}, v \in \overline{V}(\mathcal{V})\}$.

Proof. The first assertion follows from [92, Prop. 1.5] and [13, Thm. 3.1(d)] while the second one is a consequence of [13, Thm. 3.1(c)]. \square

Lemma 11.3.2. *Let X be a locally compact Hausdorff space and let w be a positive function on X . Let $\mathcal{V} = (v_n)_n$ be a decreasing weight system such that*

$$\forall n \exists m \geq n \exists C > 0 \forall x \in X : v_m(x) \leq Cw(x)v_n(x).$$

Then,

$$\forall v \in \overline{V} \exists \bar{v} \in \overline{V} \forall x \in X : v(x) \leq w(x)\bar{v}(x).$$

Proof. Let $(n_j)_j$ be an increasing sequence of natural numbers such that $v_{n_{j+1}} \leq C_j w v_{n_j}$ for all $j \in \mathbb{N}$ and some $C_j > 0$. Next, choose $C'_j > 0$ such that $v \leq C'_j v_{n_j}$ for all $j \in \mathbb{N}$. Set $\bar{v} = \inf_j C_j C'_{j+1} v_{n_j} \in \overline{V}$. Then,

$$v \leq \inf_{j \in \mathbb{N}} C'_{j+1} v_{n_{j+1}} \leq w \inf_{j \in \mathbb{N}} C_j C'_{j+1} v_{n_j} = w\bar{v}.$$

\square

We are ready to establish the mapping properties of the STFT. We define $\mathcal{V}_{(M_p)} := (e^{-M(\cdot)})_n$ and $\mathcal{W}_{\{M_p\}} := (e^{-M(\cdot/N)})_N$, a decreasing and an increasing weight system on \mathbb{R}^d , respectively.

Proposition 11.3.3. *Let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$.*

(i) *The mappings*

$$V_\psi : \mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{V}C) \rightarrow \mathcal{V} \otimes \mathcal{V}_{(M_p)} C(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi^* : \mathcal{V} \otimes \mathcal{V}_{(M_p)} C(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{V}C)$$

are well-defined and continuous.

(ii) *The mappings*

$$V_\psi : \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{V}C) \rightarrow \mathcal{V}C(\mathbb{R}_x^d) \widehat{\otimes}_\varepsilon \mathcal{W}_{\{M_p\}} C(\mathbb{R}_\xi^d)$$

and

$$V_\psi^* : \mathcal{V}C(\mathbb{R}_x^d) \widehat{\otimes}_\varepsilon \mathcal{W}_{\{M_p\}} C(\mathbb{R}_\xi^d) \rightarrow \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{V}C)$$

are well-defined and continuous.

Proof. (i) We first consider V_ψ . Let $f \in \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{V}C)$ and $v \in \overline{V}(\mathcal{V} \otimes \mathcal{V}_{(M_p)})$ be arbitrary. By Lemma 9.1.6 there are $v_1 \in \overline{V}(\mathcal{V})$ and $v_2 \in \overline{V}(\mathcal{V}_{(M_p)})$ such that $v \leq v_1 \otimes v_2$. Notice that the set $B = \{M_\xi \check{\psi} v_2(\xi) \mid \xi \in \mathbb{R}^d\}$ is bounded in $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ and that

$$\begin{aligned} \|V_\psi f\|_{Cv} &\leq \sup_{(x, \xi) \in \mathbb{R}^{2d}} |(f * M_\xi \check{\psi})(x)| v_1(x) v_2(\xi) \\ &\leq \sup_{\varphi \in B} \|f * \varphi\|_{Cv_1}, \end{aligned}$$

whence V_ψ is well-defined and continuous. Next, we treat V_ψ^* . It suffices to show that for all $n \in \mathbb{N}$ there is $m \geq n$ such that for all $B \subset \mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ bounded there is $C > 0$ such that $\sup_{\varphi \in B} \|V_\psi F * \varphi\|_{Cv_m} \leq C \|F\|_{Cv_n \otimes e^{-M(n \cdot)}}$ for all $F \in Cv_n \otimes e^{-M(n \cdot)}(\mathbb{R}_{x, \xi}^{2d})$. Let $m \geq n$ be such that $v_m/v_n \in L^1(\mathbb{R}^d)$ and choose $\tilde{m} \geq m$ such that $v_{\tilde{m}}(x + y) \leq Cv_m(x)e^{A(y/\lambda)}$ for all $x, y \in \mathbb{R}^d$ and some $C, \lambda > 0$. Proposition 9.3.7 implies that

$$\begin{aligned} &\sup_{\varphi \in B} \|V_\psi^* F * \varphi\|_{Cv_{\tilde{m}}} \\ &\leq \sup_{\varphi \in B} \sup_{t \in \mathbb{R}^d} v_{\tilde{m}}(t) \int \int_{\mathbb{R}^{2d}} |F(x, \xi)| |V_\psi \check{\varphi}(x - t, \xi)| dx d\xi \\ &\leq C \sup_{\varphi \in B} \sup_{t \in \mathbb{R}^d} \int \int_{\mathbb{R}^{2d}} |F(x, \xi)| v_m(x) |V_\psi \check{\varphi}(x - t, \xi)| e^{A((x-t)/\lambda)} dx d\xi \\ &\leq C' \|F\|_{Cv_n \otimes e^{-M(n \cdot)}} \end{aligned}$$

for all $F \in Cv_n \otimes e^{-M(n \cdot)}(\mathbb{R}_{x, \xi}^{2d})$, where the constant C' is given by

$$C_0 C \sup_{\varphi \in B} \|V_\psi \check{\varphi}\|_{Ce^{A(\cdot/\lambda)} \otimes e^{M(Hn \cdot)}} \int_{\mathbb{R}^d} \frac{v_m(x)}{v_n(x)} dx \int_{\mathbb{R}^d} e^{-M(n\xi)} d\xi < \infty.$$

(ii) We first consider V_ψ . Let $f \in \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{VC})$, $N \in \mathbb{N}$, and $v \in \overline{V}(\mathcal{V})$ be arbitrary. Notice that the set $B = \{M_\xi \check{\psi} e^{-M(\xi/N)} \mid \xi \in \mathbb{R}^d\}$ is bounded in $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ and that

$$\begin{aligned} \|V_\psi f\|_{Cv \otimes e^{-M(\cdot/N)}} &\leq \sup_{(x, \xi) \in \mathbb{R}^{2d}} |(f * M_\xi \check{\psi})(x)| v(x) e^{-M(\xi/N)} \\ &\leq \sup_{\varphi \in B} \|f * \varphi\|_{Cv}, \end{aligned}$$

whence V_ψ is well-defined and continuous. Next, we treat V_ψ^* . Let $v \in \overline{V}(\mathcal{V})$ and $B \subset \mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ bounded be arbitrary. Proposition 9.3.7 implies that there are $C, h, \lambda > 0$ such that $|V_\psi \check{\varphi}(x, \xi)| \leq C e^{-A(x/\lambda) - M(\xi/h)}$, $(x, \xi) \in \mathbb{R}^{2d}$, for all $\varphi \in B$. Next, by Lemma 11.2.17, there is $\bar{v} \in \overline{V}(\mathcal{V})$ such that $v(x + y) \leq C' \bar{v}(x) e^{A(y/\lambda)}$ for all $x, y \in \mathbb{R}^d$ and some $C' > 0$ while, by Lemma 11.3.2, there is $\bar{\bar{v}} \in \overline{V}(\mathcal{V})$ such that $\bar{v}/\bar{\bar{v}} \in L^1(\mathbb{R}^d)$. Hence,

$$\begin{aligned} &\sup_{\varphi \in B} \|V_\psi^* F * \varphi\|_{Cv} \\ &\leq C' \sup_{\varphi \in B} \sup_{t \in \mathbb{R}^d} \int \int_{\mathbb{R}^{2d}} |F(x, \xi)| \bar{v}(x) |V_\psi \check{\varphi}(x - t, \xi)| e^{A((x-t)/\lambda)} dx d\xi \\ &\leq C'' \|F\|_{C\bar{\bar{v}} \otimes e^{-M(\cdot/(Hh))}} \end{aligned}$$

for all $F \in \mathcal{VC}(\mathbb{R}_x^d) \widehat{\otimes}_\varepsilon \mathcal{W}_{\{M_p\}} C(\mathbb{R}_\xi^d)$, where

$$C'' = C_0 C C' \int_{\mathbb{R}^d} \frac{\bar{v}(x)}{\bar{\bar{v}}(x)} dx \int_{\mathbb{R}^d} e^{-M(\xi/(Hh))} d\xi < \infty.$$

□

Proposition 11.3.3 and (9.3.6) imply that:

Corollary 11.3.4. *The space $\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{VC})$ is complete.*

Proposition 11.3.5. *Let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$.*

(i) *The mappings*

$$V_\psi : \mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{WC}) \rightarrow \mathcal{WC}(\mathbb{R}_x^d) \widehat{\otimes}_\varepsilon \mathcal{V}_{(M_p)} C(\mathbb{R}_\xi^d)$$

and

$$V_\psi^* : \mathcal{WC}(\mathbb{R}_x^d) \widehat{\otimes}_\varepsilon \mathcal{V}_{(M_p)} C(\mathbb{R}_\xi^d) \rightarrow \mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{WC})$$

are well-defined and continuous.

(ii) The mappings

$$V_\psi : \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{WC}) \rightarrow \mathcal{W} \otimes \mathcal{W}_{\{M_p\}} C(\mathbb{R}_{x,\xi}^{2d})$$

and

$$V_\psi^* : \mathcal{W} \otimes \mathcal{W}_{\{M_p\}} C(\mathbb{R}_{x,\xi}^{2d}) \rightarrow \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{WC})$$

are well-defined and continuous.

Proof. This can be shown in a similar way as Proposition 11.3.3. \square

Proposition 11.3.5 and (9.3.6) imply that:

Corollary 11.3.6. *The space $\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{WC})$ is complete.*

11.3.2 Preduals

Next, we determine a predual of $\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{VC})$ and $\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{WC})$. We start with the following lemma whose verification is left to the reader.

Lemma 11.3.7. *Let v and w be non-negative functions on \mathbb{R}^d satisfying (11.2.1). Then,*

$$\|T_x \varphi\|_{\mathcal{B}_v^{M_p, h}} \leq C w(x) \|\varphi\|_{\mathcal{S}_{A_p, \lambda}^{M_p, h}}, \quad \varphi \in \mathcal{S}_{A_p, \lambda}^{M_p, h}(\mathbb{R}^d).$$

Theorem 11.3.8.

$$(i) \quad (\mathcal{B}_{\mathcal{V}^\circ}^{(M_p)})'_b = \mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{VC}).$$

$$(ii) \quad (\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}})'_{bs} = \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{VC}).$$

Proof. (i) Lemma 11.3.7 yields that $(\mathcal{B}_{\mathcal{V}^\circ}^{(M_p)}(\mathbb{R}^d))' \subseteq \mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{V}C)$. We now show that this inclusion holds continuously if we endow the former space with the strong topology. Let $v \in \overline{V}(\mathcal{V})$ and $B \subset \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ bounded be arbitrary. The set $B' = \{T_x \check{\varphi} v(x) \mid x \in \mathbb{R}^d, \varphi \in B\}$ is bounded in $\mathcal{B}_{\mathcal{V}^\circ}^{(M_p)}$ and we have that

$$\sup_{\varphi \in B} \|f * \varphi\|_{Cv} = \sup_{\varphi \in B} \sup_{x \in \mathbb{R}^d} |\langle f, T_x \check{\varphi} \rangle| v(x) = \sup_{\chi \in B'} |\langle f, \chi \rangle|$$

for all $f \in (\mathcal{B}_{\mathcal{V}^\circ}^{(M_p)})'$. Next, we show that $\mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{V}C)$ is continuously included in $(\mathcal{B}_{\mathcal{V}^\circ}^{(M_p)})'_b$. By Proposition 11.3.3(i) and (9.3.6) it suffices to show that

$$V_\psi^* : \mathcal{V} \otimes \mathcal{V}_{(M_p)} C(\mathbb{R}_{x,\xi}^{2d}) \rightarrow (\mathcal{B}_{\mathcal{V}^\circ}^{(M_p)}(\mathbb{R}^d))'_b$$

is well-defined continuous mapping. Let $F \in \mathcal{V} \otimes \mathcal{V}_{(M_p)} C(\mathbb{R}_{x,\xi}^{2d})$ be arbitrary. The linear functional

$$f : \mathcal{B}_{\mathcal{V}^\circ}^{(M_p)}(\mathbb{R}^d) \rightarrow \mathbb{C} : \varphi \rightarrow \int \int_{\mathbb{R}^{2d}} F(x, \xi) V_{\overline{\psi}} \varphi(x, -\xi) dx d\xi$$

is well-defined and continuous by Proposition 11.2.13(i). Since $V_\psi^* F = f|_{\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)}$, we obtain that $V_\psi^* F \in (\mathcal{B}_{\mathcal{V}^\circ}^{(M_p)}(\mathbb{R}^d))'$ and

$$\langle V_\psi^* F, \varphi \rangle = \int \int_{\mathbb{R}^{2d}} F(x, \xi) V_{\overline{\psi}} \varphi(x, -\xi) dx d\xi$$

for all $\varphi \in \mathcal{B}_{\mathcal{V}^\circ}^{(M_p)}(\mathbb{R}^d)$. Finally, we show that V_ψ^* is continuous. Let $B \subset \mathcal{B}_{\mathcal{V}^\circ}^{(M_p)}(\mathbb{R}^d)$ bounded be arbitrary. By Proposition 11.2.13(i) and Lemma 9.1.6 there are $v_1 \in \overline{V}(\mathcal{V})$ and $v_2 \in \overline{V}(\mathcal{V}_{(M_p)})$ such that $|V_{\overline{\psi}} \varphi(x, -\xi)| \leq v_1(x) v_2(\xi)$, $(x, \xi) \in \mathbb{R}^{2d}$, for all $\varphi \in B$. Lemma 11.3.2 implies that there is $\bar{v}_1 \in \overline{V}(\mathcal{V})$ such that $v_1/\bar{v}_1 \in L^1(\mathbb{R}^d)$. Set $\bar{v}_2 = v_2(\cdot)(1 + |\cdot|)^{d+1} \in \overline{V}(\mathcal{V}_{(M_p)})$. Then,

$$\begin{aligned} \sup_{\varphi \in B} |\langle V_\psi^* F, \varphi \rangle| &\leq \sup_{\varphi \in B} \int \int_{\mathbb{R}^{2d}} |F(x, \xi)| |V_{\overline{\psi}} \varphi(x, -\xi)| dx d\xi \\ &\leq \int \int |F(x, \xi)| v_1(x) v_2(\xi) dx d\xi \leq C \|F\|_{C\bar{v}_1 \otimes \bar{v}_2} \end{aligned}$$

for all $F \in \mathcal{V} \otimes \mathcal{V}_{(M_p)} C(\mathbb{R}_{x,\xi}^{2d})$, where

$$C = \int_{\mathbb{R}^d} \frac{v_1(x)}{\bar{v}_1(x)} dx \int_{\mathbb{R}^d} (1 + |\xi|)^{-(d+1)} d\xi < \infty.$$

(ii) Lemma 11.3.7 implies that $(\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))' \subseteq \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{V}C)$. We now show that this inclusion holds continuously if we endow the former space with the bs -topology. Let $v \in \bar{V}(\mathcal{V})$ and $B \subset \mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$ bounded be arbitrary. There is $h > 0$ such that B is contained and bounded in $\mathcal{S}_{A_p,h}^{M_p,h}(\mathbb{R}^d)$. The set $B' = \{T_x \check{\varphi} v(x) \mid x \in \mathbb{R}^d, \varphi \in B\}$ is bounded in $\mathcal{B}_{\mathcal{V}^\circ}^{M_p,h}$ and we have that

$$\sup_{\varphi \in B} \|f * \varphi\|_{Cv} = \sup_{\varphi \in B} \sup_{x \in \mathbb{R}^d} |\langle f, T_x \check{\varphi} \rangle| v(x) = \sup_{\chi \in B'} |\langle f, \chi \rangle|$$

for all $f \in (\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}})'$. Next, we show that $\mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{V}C)$ is continuously included in $(\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}})'_{bs}$. By Proposition 11.3.3(ii) and (9.3.6) it suffices to show that

$$V_\psi^* : \mathcal{V}C(\mathbb{R}_x^d) \widehat{\otimes}_\varepsilon \mathcal{W}_{\{M_p\}} C(\mathbb{R}_\xi^d) \rightarrow (\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))'_{bs}$$

is well-defined and continuous. Let $F \in \mathcal{V}C(\mathbb{R}_x^d) \widehat{\otimes}_\varepsilon \mathcal{W}_{\{M_p\}} C(\mathbb{R}_\xi^d)$ be arbitrary. The linear functional

$$f : \mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d) \rightarrow \mathbb{C} : \varphi \rightarrow \int \int_{\mathbb{R}^{2d}} F(x, \xi) V_{\check{\psi}} \varphi(x, -\xi) dx d\xi$$

is well-defined and continuous by Lemma 11.2.11. Since $V_\psi^* F = f|_{\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)}$, we obtain that $V_\psi^* F \in (\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))'$ and

$$\langle V_\psi^* F, \varphi \rangle = \int \int_{\mathbb{R}^{2d}} F(x, \xi) V_{\check{\psi}} \varphi(x, -\xi) dx d\xi$$

for all $\varphi \in \mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d)$. Finally, we show that V_ψ^* is continuous. Let $h > 0$ and $B \subset \mathcal{B}_{\mathcal{V}^\circ}^{M_p,h}(\mathbb{R}^d)$ bounded be arbitrary. By Lemma 11.2.11 there is $v \in \bar{V}(\mathcal{V})$ such that $|V_{\check{\psi}} \varphi(x, -\xi)| \leq v(x) e^{-M(\pi\xi/(\sqrt{d}h))}$, $(x, \xi) \in \mathbb{R}^{2d}$, for all $\varphi \in B$. Lemma 11.3.2 implies that there is

$\bar{v} \in \overline{V}(\mathcal{V})$ such that $v/\bar{v} \in L^1(\mathbb{R}^d)$. Then,

$$\begin{aligned} \sup_{\varphi \in B} |\langle V_\psi^* F, \varphi \rangle| &\leq \sup_{\varphi \in B} \int \int_{\mathbb{R}^{2d}} |F(x, \xi)| |V_{\bar{\psi}} \varphi(x, -\xi)| dx d\xi \\ &\leq \int \int_{\mathbb{R}^{2d}} |F(x, \xi)| v(x) e^{-M(\pi\xi/(\sqrt{d}h))} dx d\xi \leq C \|F\|_{C\bar{v} \otimes e^{-M(\pi \cdot/(\sqrt{d}Hh))}} \end{aligned}$$

for all $F \in \mathcal{V}C(\mathbb{R}_x^d) \hat{\otimes}_\varepsilon \mathcal{W}_{\{M_p\}} C(\mathbb{R}_\xi^d)$, where

$$C = C_0 \int_{\mathbb{R}^d} \frac{v(x)}{\bar{v}(x)} dx \int_{\mathbb{R}^d} e^{-M(\pi\xi/(\sqrt{d}Hh))} d\xi < \infty.$$

□

Theorem 11.3.9.

$$(i) \quad (\mathcal{B}_{\mathcal{W}^\circ}^{(M_p)})'_{bs} = \mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{W}C).$$

$$(ii) \quad (\mathcal{B}_{\mathcal{W}^\circ}^{\{M_p\}})'_b = \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{W}C).$$

Proof. This can be shown in a similar way as Theorem 11.3.8. □

We point out the following corollary.

Corollary 11.3.10. *Let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \setminus \{0\}$ and let $\gamma \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ be a synthesis window for ψ . Then, the desingularization formula*

$$\langle f, \varphi \rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi f(x, \xi) V_{\bar{\gamma}} \varphi(x, -\xi) dx d\xi$$

holds for all $f \in (\mathcal{B}_{\mathcal{V}^\circ}^(\mathbb{R}^d))'$ and $\varphi \in \mathcal{B}_{\mathcal{V}^\circ}^*(\mathbb{R}^d)$ ($f \in (\mathcal{B}_{\mathcal{W}^\circ}^*(\mathbb{R}^d))'$ and $\varphi \in \mathcal{B}_{\mathcal{W}^\circ}^*(\mathbb{R}^d)$, respectively).*

Finally, we make two interesting observations.

Proposition 11.3.11. *Let $f \in \mathcal{S}_\dagger^*(\mathbb{R}^d)$. Then, $f \in \mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{V}C)$ if and only if $f * \varphi \in \mathcal{B}_\mathcal{V}^*(\mathbb{R}^d)$ for all $\varphi \in \mathcal{S}_\dagger^*(\mathbb{R}^d)$. Moreover, the topology of $\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{V}C)$ coincides with the initial topology with respect to the mapping*

$$\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{V}C) \rightarrow L_b(\mathcal{S}_\dagger^*(\mathbb{R}^d), \mathcal{B}_\mathcal{V}^*(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi).$$

Proof. We write $\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{B}_\mathcal{V}^*)$ for the space consisting of all $f \in \mathcal{S}_\dagger^*(\mathbb{R}^d)$ such that $f * \varphi \in \mathcal{B}_\mathcal{V}^*(\mathbb{R}^d)$ for all $\varphi \in \mathcal{S}_\dagger^*(\mathbb{R}^d)$ and endow it with the initial topology with respect to the mapping

$$\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{B}_\mathcal{V}^*) \rightarrow L_b(\mathcal{S}_\dagger^*(\mathbb{R}^d), \mathcal{B}_\mathcal{V}^*(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi).$$

Obviously, $\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{B}_\mathcal{V}^*) \subseteq \mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{V}C)$ with continuous inclusion. For the converse we employ Theorem 11.3.8.

Beurling case. Since $\mathcal{B}_{\mathcal{V}^\circ}^{(M_p)}$ is an (FS) -space (see Proposition 11.2.2), it suffices to show that every bounded set $B \subset (\mathcal{B}_{\mathcal{V}^\circ}^{(M_p)})'_b$ is contained and bounded in $\mathcal{O}'_C(\mathcal{S}_\dagger^*, \mathcal{B}_\mathcal{V}^*)$. Let $n \in \mathbb{N}$ and $C, h > 0$ be such that

$$|\langle f, \varphi \rangle| \leq C \|\varphi\|_{\mathcal{B}_{1/v_n}^{M_p, h}}, \quad \varphi \in \mathcal{B}_{\mathcal{V}^\circ}^{(M_p)}(\mathbb{R}^d),$$

for all $f \in B$. Choose $m \geq n$ such that $v_m(x + y) \leq C' v_n(x) e^{A(y/\lambda)}$ for all $x, y \in \mathbb{R}^d$ and some $C', \lambda > 0$. Let $k > 0$ be arbitrary. Lemma 11.3.7 implies that

$$\begin{aligned} \|f * \varphi\|_{\mathcal{B}_{1/v_m}^{M_p, k}} &\leq C \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{\|T_x(\partial^\alpha \check{\varphi})\|_{\mathcal{B}_{1/v_n}^{M_p, h}}}{k^{|\alpha|} M_\alpha v_m(x)} \\ &\leq CC' \sup_{\alpha \in \mathbb{N}^d} \frac{\|\partial^\alpha \check{\varphi}\|_{\mathcal{S}_{A_p, \lambda}^{M_p, h}}}{k^{|\alpha|} M_\alpha} \leq C_0 CC' \|\check{\varphi}\|_{\mathcal{S}_{A_p, \lambda}^{M_p, \min\{k, h\}/H}} \end{aligned}$$

for all $f \in B$ and $\varphi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$, whence the result follows.

Roumieu case. By Theorem 11.2.15 it suffices to show that for every bounded set $B \subset \mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$, every $h_j \in \mathfrak{A}$ and every $v \in \overline{V}(\mathcal{V})$, there are $h > 0$ and a bounded set $B' \subset \mathcal{B}_{\mathcal{V}^\circ}^{M_p, h}(\mathbb{R}^d)$ such that

$$\sup_{\varphi \in B} \|f * \varphi\|_{\mathcal{B}_v^{M_p, h_j}} \leq \sup_{\varphi \in B'} |\langle f, \varphi \rangle|,$$

for all $f \in (\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))'$. Let $k > 0$ be such that B is contained and bounded in $\mathcal{S}_{A_p, k}^{M_p, k}(\mathbb{R}^d)$. Lemma 11.3.7 implies that

$$B' = \left\{ \frac{T_x(\partial^\alpha \check{\varphi})v(x)}{M_\alpha \prod_{j=0}^{|\alpha|} h_j} \mid \varphi \in B, \alpha \in \mathbb{N}^d, x \in \mathbb{R}^d \right\}$$

is bounded in $\mathcal{B}_{\mathcal{V}^\circ}^{M_p, kH}(\mathbb{R}^d)$. Moreover,

$$\sup_{\varphi \in B} \|f * \varphi\|_{\mathcal{B}_v^{M_p, h_j}} \leq \sup_{\varphi \in B} \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\langle f, T_x(\partial^\alpha \check{\varphi}) \rangle| v(x)}{M_\alpha \prod_{j=0}^{|\alpha|} h_j} = \sup_{\chi \in B'} |\langle f, \chi \rangle|.$$

□

Proposition 11.3.12. *Let $f \in \mathcal{S}'_+(\mathbb{R}^d)$. Then, $f \in \mathcal{O}'_C(\mathcal{S}^*_+, \mathcal{WC})$ if and only if $f * \varphi \in \mathcal{B}^*_{\mathcal{W}}(\mathbb{R}^d)$ for all $\varphi \in \mathcal{S}^*_+(\mathbb{R}^d)$. Moreover, the topology of $\mathcal{O}'_C(\mathcal{S}^*_+, \mathcal{WC})$ coincides with the initial topology with respect to the mapping*

$$\mathcal{O}'_C(\mathcal{S}^*_+, \mathcal{WC}) \rightarrow L_b(\mathcal{S}^*_+(\mathbb{R}^d), \mathcal{B}^*_{\mathcal{W}}(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi).$$

Proof. This can be shown in a similar way as Proposition 11.3.11.

□

11.3.3 Topological properties

We now discuss the locally convex structure of $\mathcal{O}'_C(\mathcal{S}^{\{M_p\}}_{\{A_p\}}, \mathcal{VC})$ and $\mathcal{O}'_C(\mathcal{S}^{(M_p)}_{(A_p)}, \mathcal{WC})$.

Lemma 11.3.13. *$\mathcal{O}'_C(\mathcal{S}^{\{M_p\}}_{\{A_p\}}, \mathcal{VC})$ and $\mathcal{O}'_C(\mathcal{S}^{(M_p)}_{(A_p)}, \mathcal{WC})$ are (PLS) -spaces.*

Proof. The space $\mathcal{O}'_C(\mathcal{S}^{\{M_p\}}_{\{A_p\}}, \mathcal{VC})$ is topologically isomorphic to a closed subspace of $L_b(\mathcal{S}^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d), \mathcal{B}^{\{M_p\}}_{\mathcal{V}}(\mathbb{R}^d))$ by Corollary 11.3.4 and Proposition 11.3.11. Similarly, Corollary 11.3.6 and Proposition 11.3.12 imply that the space $\mathcal{O}'_C(\mathcal{S}^{(M_p)}_{(A_p)}, \mathcal{WC})$ is topologically isomorphic to a closed subspace of $L_b(\mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d), \mathcal{B}^{(M_p)}_{\mathcal{W}}(\mathbb{R}^d))$. Since the class of (PLS) -spaces is closed under taking closed subspaces, we only need to show that $L_b(\mathcal{S}^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d), \mathcal{B}^{\{M_p\}}_{\mathcal{V}}(\mathbb{R}^d))$ and $L_b(\mathcal{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d), \mathcal{B}^{(M_p)}_{\mathcal{W}}(\mathbb{R}^d))$ are (PLS) -spaces. This is a consequence of Propositions 11.2.2 and 11.2.3 and the general fact that $L_b(E, F)$ is a (PLS) -space if both E and F are (FS) - or (DFS) -spaces [56, Prop. 4.3].

□

Next, we determine the strong duals of the convolutor spaces. We need two lemmas in preparation.

Lemma 11.3.14. $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ is densely and continuously included in $\mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{VC})$.

Proof. It suffices to show that $\mathcal{B}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d)$ is dense in $\mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{VC})$ by Corollary 11.2.5. Let $f \in \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{VC})$ be arbitrary. Choose $\chi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \chi(x) dx = 1$ and set $\chi_k = k^d \chi(k \cdot)$, $k \geq 1$. Define $f_k = f * \chi_k \in \mathcal{B}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d)$. Similarly as in Lemma 10.3.7, one can now show that $f_k \rightarrow f$ in $\mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{VC})$. We leave the details to the reader. \square

Lemma 11.3.15. $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ is densely and continuously included in $\mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{WC})$.

Proof. This can be shown in a similar way as Lemma 11.3.14. \square

Proposition 11.3.16. *The mapping*

$$\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d) \rightarrow ((\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))'_{bs})'_b : \varphi \rightarrow (f \rightarrow \langle f, \varphi \rangle) \quad (11.3.3)$$

is a topological isomorphism.

Proof. The bs -topology is coarser than the strong topology and finer than the weak-* topology on $(\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))'$. Since $\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d)$ is barrelled (as it is an (LF) -space), a subset of $(\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))'$ is therefore equicontinuous if and only if it is bs -bounded, which in turn yields that the mapping (11.3.3) is a strict morphism. We now show that it is surjective. Let $\Phi \in ((\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))'_{bs})'$ be arbitrary. Denote by $\iota : \mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d) \rightarrow \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{VC})$ the canonical inclusion and set $g = \Phi \circ \iota \in \mathcal{S}'_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$. Let $\psi \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) \setminus \{0\}$ and let $\gamma \in \mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ be a synthesis window for ψ . By (9.3.7) it holds that

$$\Phi(\iota(\chi)) = \langle g, \chi \rangle = \frac{1}{(\gamma, \psi)_{L^2}} \int \int_{\mathbb{R}^{2d}} V_\psi \chi(x, \xi) V_{\bar{\gamma}} g(x, -\xi) dx d\xi$$

for all $\chi \in \mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$. By Lemma 11.3.14 it therefore suffices to show that $g \in \mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d)$ or, thus, that $V_\theta g \in \mathcal{V}^\circ \otimes_i \mathcal{V}_{\{M_p\}} C(\mathbb{R}_{x,\xi}^{2d})$, where $\theta \in \mathcal{S}_{(A_p)}^{\{M_p\}}(\mathbb{R}^d)$ (Proposition 11.2.13 and (9.3.6)). Since Φ is continuous, there is $h > 0$ and a bounded set $B \subset \mathcal{B}_{\mathcal{V}^\circ}^{M_p,h}(\mathbb{R}^d)$ such that

$$\begin{aligned} |V_\theta g(x, \xi)| &= |\Phi(\iota(\overline{M_\xi T_x \theta}))| \\ &\leq \sup_{\varphi \in B} \left| \int_{\mathbb{R}^d} \varphi(t) \overline{M_\xi T_x \theta(t)} dt \right| = \sup_{\varphi \in B} |V_\theta \varphi(x, \xi)|. \end{aligned}$$

The required bounds for $|V_\theta g|$ therefore directly follow from Lemma 11.2.11. \square

Proposition 11.3.17. *The mapping*

$$\mathcal{B}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d) \rightarrow ((\mathcal{B}_{\mathcal{V}}^{\{M_p\}}(\mathbb{R}^d))'_{bs})'_b : \varphi \rightarrow (f \rightarrow \langle f, \varphi \rangle)$$

is a topological isomorphism.

Proof. This can be shown in a similar way as Proposition 11.3.16. \square

We are ready to show the main results of this section.

Theorem 11.3.18. *Consider the following conditions:*

- (i) \mathcal{V}° satisfies (DN).
- (ii) The (LFS)-space $\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d)$ satisfies one of the equivalent conditions of Corollary 2.3.2.
- (iii) $\mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{B}_{\mathcal{V}}^{\{M_p\}})$ is ultrabornological.
- (iv) $(\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))'_b = \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{V}C)$.

Then, (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (ii). Moreover, if M_p satisfies (M.1), (M.2), and (M.3), then (ii) \Rightarrow (i).

Proof. The fact that $\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d)$ is an (LFS) -space was shown in Proposition 11.2.2. Hence, $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ (under the additional assumption that M_p satisfies $(M.1)$, $(M.2)$, and $(M.3)$) follow from Theorem 11.2.7. We now show the other implications; recall that $(\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))'_{bs} = \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{V}C)$ (Theorem 11.3.8).

$(ii) \Rightarrow (iii)$ Notice that $(\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d))'_b = \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{V}C)$ because $\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d)$ is regular, whence the result follows from the general fact that the strong dual of a complete Schwartz space is ultrabornological [150, p. 43].

$(iii) \Rightarrow (iv)$ The strong topology on $(\mathcal{B}_{\mathcal{W}}^{\{M_p\}}(\mathbb{R}^d))'$ is finer than the bs -topology. As the strong dual of an (LF) -space is strictly webbed [49, Prop. IV.3.3], they are identical by De Wilde's open mapping theorem.

$(iv) \Rightarrow (ii)$ Proposition 11.3.16 yields that $\mathcal{B}_{\mathcal{V}^\circ}^{\{M_p\}}(\mathbb{R}^d)$ is reflexive and, thus, quasi-complete. \square

Remark 11.3.19. One can give more direct (and perhaps also more insightful) proofs of the various implications in Theorem 11.3.18 by using the same ideas as in the proof of Theorem 10.3.12.

Theorem 11.3.20. *Consider the following conditions:*

- (i) \mathcal{W}° satisfies (Ω) .
- (ii) The (LFS) -space $\mathcal{B}_{\mathcal{W}^\circ}^{(M_p)}(\mathbb{R}^d)$ satisfies one of the equivalent conditions of Corollary 2.3.2.
- (iii) $\mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{W}C)$ is ultrabornological.
- (iv) $(\mathcal{B}_{\mathcal{W}^\circ}^{(M_p)}(\mathbb{R}^d))'_b = \mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{W}C)$.

Then, $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (ii)$. Moreover, if M_p satisfies $(M.1)$, $(M.2)$, and $(M.3)$, then $(ii) \Rightarrow (i)$.

Proof. This can be shown in a similar way as Theorem 11.3.18. \square

11.4 Applications

We now apply the results from Sections 11.2 and 11.3 to study several convolution properties of the space $\mathcal{S}'_{\dagger}(\mathbb{R}^d)$. Unless explicitly stated otherwise, M_p and A_p will stand for a pair of weight sequences satisfying (9.2).

Notice that

$$\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d) = \mathcal{B}_{\mathcal{W}_{(A_p)}}^{(M_p)}(\mathbb{R}^d), \quad \mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d) = \mathcal{B}_{\mathcal{V}_{\{A_p\}}}^{\{M_p\}}(\mathbb{R}^d)$$

while we define

$$\mathcal{O}_C^{(M_p), (A_p)}(\mathbb{R}^d) := \mathcal{B}_{\mathcal{V}_{(A_p)}}^{(M_p)}(\mathbb{R}^d), \quad \mathcal{O}_C^{\{M_p\}, \{A_p\}}(\mathbb{R}^d) := \mathcal{B}_{\mathcal{W}_{\{A_p\}}}^{\{M_p\}}(\mathbb{R}^d).$$

Remark 11.4.1. Analogues of the space $\mathcal{O}_C(\mathbb{R}^d)$ in the setting of the Gelfand-Shilov spaces $\mathcal{S}'_{\dagger}(\mathbb{R}^d)$ have been studied in [50]. In the Beurling case, our space $\mathcal{O}_C^{(M_p), (M_p)}(\mathbb{R}^d)$ coincides with $\mathcal{O}_C^{(M_p)}(\mathbb{R}^d)$ from [50] while $\mathcal{O}_C^{\{M_p\}, \{M_p\}}(\mathbb{R}^d)$ differs from the one denoted by $\mathcal{O}_C^{\{M_p\}}(\mathbb{R}^d)$ there; see also Remark 11.4.5 below.

11.4.1 Characterization of tempered ultradistribution spaces via convolution averages

Let B_p be a weight sequence. We employ the notation $\dagger = (B_p)$ or $\{B_p\}$ to treat the Beurling and Roumieu case simultaneously. If B_p satisfies $(M.2)'$, then \mathcal{W}_{\dagger} and \mathcal{V}_{\dagger} satisfy (9.1.2) and (9.1.1), respectively. If, in addition, $A_p \subset B_p$, then \mathcal{W}_{\dagger} and \mathcal{V}_{\dagger} are both \dagger -admissible. Theorems 11.3.8 and 11.3.9 and Propositions 11.3.11 and 11.3.12 imply the following result.

Theorem 11.4.2. *Let B_p be a weight sequence satisfying $(M.2)'$ such that $A_p \subset B_p$. For $f \in \mathcal{S}'_{\dagger}(\mathbb{R}^d)$ the following statements are equivalent:*

$$(i) \quad f \in \mathcal{S}'_{\dagger}(\mathbb{R}^d).$$

$$(ii) \quad f * \varphi \in \mathcal{V}_{(B_p)}C(\mathbb{R}^d) \quad (f * \varphi \in \mathcal{W}_{\{B_p\}}C(\mathbb{R}^d)) \text{ for all } \varphi \in \mathcal{S}'_{\dagger}(\mathbb{R}^d).$$

$$(iii) \quad f * \varphi \in \mathcal{O}_C^{*,\dagger}(\mathbb{R}^d) \text{ for all } \varphi \in \mathcal{S}'_{\dagger}(\mathbb{R}^d).$$

Moreover, the strong topology on $\mathcal{S}'_{\dagger}(\mathbb{R}^d)$ coincides with the initial topology with respect to the mapping

$$\begin{aligned} \mathcal{S}'_{(B_p)}(\mathbb{R}^d) &\rightarrow L_b(\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d), \mathcal{V}_{(B_p)}C(\mathbb{R}^d)) : \varphi \rightarrow (f \rightarrow f * \varphi) \\ (\mathcal{S}'_{\{B_p\}}(\mathbb{R}^d) &\rightarrow L_b(\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d), \mathcal{W}_{\{B_p\}}C(\mathbb{R}^d)) : \varphi \rightarrow (f \rightarrow f * \varphi)) \end{aligned}$$

and also with the initial topology with respect to the mapping

$$\mathcal{S}'_{\dagger}(\mathbb{R}^d) \rightarrow L_b(\mathcal{S}_{\dagger}^*(\mathbb{R}^d), \mathcal{O}_C^{*,\dagger}(\mathbb{R}^d)) : \varphi \rightarrow (f \rightarrow f * \varphi).$$

Remark 11.4.3. Theorem 11.4.2 was already shown in [134, Cor. 2.8] under much more restrictive conditions on M_p and A_p and via completely different methods, namely, the authors used the Schwartz parametrix method.

11.4.2 Convolution spaces of the Gelfand-Shilov spaces $\mathcal{S}_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{S}_{\{A_p\}}^{\{M_p\}}(\mathbb{R}^d)$

In this subsection, we are interested in the convolution spaces of the Gelfand-Shilov spaces $\mathcal{S}_{\dagger}^*(\mathbb{R}^d)$, that is,

$$\mathcal{O}'_C(\mathcal{S}_{\dagger}^*) := \{f \in \mathcal{S}'_{\dagger}(\mathbb{R}^d) \mid f * \varphi \in \mathcal{S}_{\dagger}^*(\mathbb{R}^d) \text{ for all } \varphi \in \mathcal{S}_{\dagger}^*(\mathbb{R}^d)\}$$

endowed with the initial topology with respect to the mapping

$$\mathcal{O}'_C(\mathcal{S}_{\dagger}^*) \rightarrow L_b(\mathcal{S}_{\dagger}^*(\mathbb{R}^d), \mathcal{S}_{\dagger}^*(\mathbb{R}^d)) : f \rightarrow (\varphi \rightarrow f * \varphi).$$

Propositions 11.3.11 and 11.3.12 imply that

$$\mathcal{O}'_C(\mathcal{S}_{(A_p)}^{(M_p)}) = \mathcal{O}_C(\mathcal{S}_{(A_p)}^{(M_p)}, \mathcal{W}_{(A_p)}), \quad \mathcal{O}'_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}) = \mathcal{O}_C(\mathcal{S}_{\{A_p\}}^{\{M_p\}}, \mathcal{V}_{\{A_p\}}).$$

Hence, Theorems 11.3.8 and 11.3.9 yield that:

Theorem 11.4.4. $(\mathcal{O}_C^{*,\dagger}(\mathbb{R}^d))'_{bs} = \mathcal{O}'_C(\mathcal{S}^*_\dagger)$.

Remark 11.4.5. The fact that $(\mathcal{O}_C^{(M_p), (A_p)}(\mathbb{R}^d))'$ and $\mathcal{O}'_C(\mathcal{S}^{(M_p)}_{(A_p)})$ coincide algebraically has essentially already been shown in [50, Thm. 3.2]. However, due to an error carried over from [52, Prop. 2], the assertion in [50, Thm. 3.2] in the Roumieu case is wrong. In fact, the space $\mathcal{O}_C^{\{M_p\}}(\mathbb{R}^d)$ defined on [50, p. 407] is the space of multipliers of $\mathcal{S}^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d)$ and, thus, not the correct analogue of $\mathcal{O}_C(\mathbb{R}^d)$.

Next, we study the topological properties of the spaces $\mathcal{O}_C^{*,\dagger}(\mathbb{R}^d)$ and $\mathcal{O}'_C(\mathcal{S}^*_\dagger)$. For this, we first need to discuss the properties of the weight systems $\mathcal{V}_{(A_p)}$ and $\mathcal{W}_{\{A_p\}}$.

Lemma 11.4.6. *Let A_p be a weight sequence satisfying (M.1).*

(i) *Assume, in addition, that A_p satisfies (M.2). Then, $\mathcal{V}_{(A_p)}$ satisfies (Ω) if and only if A_p satisfies $(M.2)^*$.*

(ii) *$\mathcal{W}_{\{A_p\}}$ satisfies (DN) if and only if*

$$\forall h > 0 \exists k > 0 \exists C > 0 \forall t \geq 0 : A(t) + A(kt) \leq 2A(ht) + C. \quad (11.4.1)$$

Proof. (i) This follows from 10.4.12 and the characterization of conditions (M.2) and $(M.2)^*$ in terms of the associated function (cf. Subsection 2.2.1).

(ii) Obvious. □

Theorems 11.2.8 and 11.2.7 therefore yield that:

Theorem 11.4.7. *Let M_p and A_p be two weight sequences satisfying (M.1).*

(i) *Assume that A_p satisfies (M.2). If A_p satisfies $(M.2)^*$, then $\mathcal{O}_C^{(M_p), (A_p)}(\mathbb{R}^d)$ is boundedly retractive. If, in addition, M_p satisfies (M.1), (M.2), and (M.3), then $\mathcal{O}_C^{(M_p), (A_p)}(\mathbb{R}^d)$ is boundedly retractive if and only if it satisfies (wQ) if and only if A_p satisfies $(M.2)^*$.*

(iii) The space $\mathcal{O}_C^{\{M_p\}, \{A_p\}}(\mathbb{R}^d)$ is boundedly retractive if A_p satisfies (11.4.1). If, in addition, M_p satisfies (M.1), (M.2), and (M.3), then $\mathcal{O}_C^{\{M_p\}, \{A_p\}}(\mathbb{R}^d)$ is boundedly retractive if and only if it satisfies (wQ) if and only if A_p satisfies (11.4.1).

Remark 11.4.8. Condition (11.4.1) is satisfied by the q -gevreys sequences $A_p = q^{p^2}$, $q > 1$. On the other hand, it is very important to point out that if A_p satisfies (M.1) and (M.2), then (11.4.1) cannot hold for A_p . For example, this is always the case for the Gevrey sequences $A_p = p^\sigma$, $\sigma > 0$. We can thus supplement Theorem 11.4.7 as follows:

Theorem 11.4.9. *Let M_p be a weight sequence satisfying (M.1), (M.2), and (M.3) and let A_p be a weight sequence satisfying (M.1) and (M.2). Then, the space $\mathcal{O}_C^{\{M_p\}, \{A_p\}}(\mathbb{R}^d)$ does not satisfy (wQ).*

Finally, we employ Theorems 11.3.20 and 11.3.18 to discuss the locally convex structure of $\mathcal{O}'_C(\mathcal{S}^*)$. In particular, the next two results settle the question posed after [50, Thm. 3.3].

Theorem 11.4.10. *Let M_p and A_p be two weight sequences satisfying (9.2). Then, $\mathcal{O}'_C(\mathcal{S}_{(M_p)}^{(A_p)})$ is ultrabornological and*

$$(\mathcal{O}_C^{(M_p), (A_p)}(\mathbb{R}^d))'_b = \mathcal{O}'_C(\mathcal{S}_{(M_p)}^{(A_p)})$$

if A_p satisfies (M.2). If, in addition, M_p satisfies (M.1), (M.2), and (M.3), then each of these conditions is equivalent to the fact that A_p satisfies (M.2)*.*

Theorem 11.4.11. *Let M_p be a weight sequence satisfying (M.1), (M.2), and (M.3) and let A_p be a weight sequence satisfying (M.1) and (M.2). Then, $\mathcal{O}'_C(\mathcal{S}_{\{M_p\}}^{\{A_p\}})$ is not ultrabornological and the topology on $\mathcal{O}'_C(\mathcal{S}_{\{M_p\}}^{\{A_p\}})$ is strictly coarser than the one induced on this space via the strong topology on $(\mathcal{O}_C^{\{M_p\}, \{A_p\}}(\mathbb{R}^d))'$.*

Part III

Hyperfunctions and ultrahyperfunctions of fast growth

Chapter 12

Introduction

The general theory of Fourier hyperfunctions was developed by Kawai [87] who constructed a flabby sheaf on the radial compactification of \mathbb{R}^d that coincides with the sheaf \mathfrak{B} of hyperfunctions on \mathbb{R}^d and whose space of global sections is given by the dual of the Gelfand-Shilov space $\mathcal{S}_{\{p!\}}^{\{p!\}}(\mathbb{R}^d)$. Furthermore, he studied infinite order PDE's with constant coefficients in this setting and obtained many important results on existence, ellipticity, and propagation of singularities. Interestingly, several basic problems in the theory of PDE's naturally lead to generalized function spaces whose elements are not (Fourier) hyperfunctions. For example, as already pointed out in the introduction of Part I, the famous Lewy equation [107] does not admit a hyperfunctional solution [146] but is solvable in the space of tempered ultrahyperfunctions [124]. The latter space and the even larger space of Fourier ultrahyperfunctions, which is defined as the dual of the Gelfand-Shilov space $\mathcal{S}_{(p!)}^{(p!)}(\mathbb{R}^d)$, were introduced in one dimension by Silva [152] and in several variables by Hasumi [75] and Park and Morimoto [128]; see also [83, 153, 115, 167]. We refer to [104, 59, 60] for modern investigations on these spaces.

In this part, we are interested in the following generalization of the work of Kawai and Silva in the one dimensional case. Let $\omega : [0, \infty) \rightarrow [0, \infty)$ be an increasing function. For $h > 0$ we denote

by T^h the horizontal strip $|\operatorname{Im} z| < h$ of the complex plane. We shall study the space $\mathcal{U}_{(\omega)}(\mathbb{C})$ of entire functions φ satisfying

$$\sup_{z \in T^h} |\varphi(z)| e^{\omega(\lambda |\operatorname{Re} z|)} < \infty \quad (12.0.1)$$

for every $h, \lambda > 0$ and the space $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ of analytic functions φ defined on the strip T^h and satisfying the estimate (12.0.1) for some $h, \lambda > 0$. We call the elements of their duals $\mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$ *ultrahyperfunctions of type (ω)* and *hyperfunctions of type $\{\omega\}$* , respectively, or, for short, ultrahyperfunctions and hyperfunctions of fast growth. When $\omega(t) = t$, one recovers the spaces of Fourier ultrahyperfunctions and Fourier hyperfunctions.

Our main objectives are to characterize the non-triviality of the test function spaces $\mathcal{U}_{(\omega)}(\mathbb{C})$ and $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ in terms of the growth order of ω and to develop an analytic representation theory for (ultra)hyperfunctions of fast growth in the spirit of Silva [152]. In particular, we shall express the dual spaces $\mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$ as quotients of spaces of analytic functions satisfying certain growth estimates with respect to ω . These results will be presented in Chapter 13. As an application of our ideas, we study in Chapter 14 boundary values of analytic functions in Beurling-Björck ultra-distribution spaces of exponential type (cf. [14]).

Chapter 13

Analytic representation theory and the non-triviality of certain spaces of analytic functions

13.1 Introduction

The first part of this chapter is devoted to a detailed study of the spaces $\mathcal{U}_{(\omega)}(\mathbb{C})$ and $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ defined in the introduction of this part (Chapter 12). Our main result asserts that $\mathcal{U}_{(\omega)}(\mathbb{C})$ contains a function that is not identically zero if and only if

$$\lim_{t \rightarrow \infty} e^{-\mu t} \omega(t) = 0 \quad (13.1.1)$$

for all $\mu > 0$. The corresponding statement for $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ holds if and only if (13.1.1) is satisfied for some $\mu > 0$, which is essentially a result due to Mandelbrojt [108, Sect. 2.1] that we shall reprove here. These characterizations are of similar nature to the Denjoy-Carleman theorem in the theory of ultradifferentiable functions. In the case of $\mathcal{A}_{\{\omega\}}(\mathbb{R})$, the result will follow from complex analytic arguments while the analysis of $\mathcal{U}_{(\omega)}(\mathbb{C})$ requires a more elaborate

treatment, involving duality theory and analytic representations. It is worth pointing out that when $\omega = M$ is the associated function of a weight sequence M_p , our test function spaces coincide with certain Gelfand-Shilov spaces, namely, $\mathcal{U}_{(M)}(\mathbb{C}) = \mathcal{S}_{(M_p)}^{(p!)}(\mathbb{R})$ and $\mathcal{A}_{\{M\}}(\mathbb{R}) = \mathcal{S}_{\{M_p\}}^{\{p!\}}(\mathbb{R})$. Specializing our result, we obtain that $\mathcal{S}_{(M_p)}^{(p!)}(\mathbb{R})$ and $\mathcal{S}_{\{M_p\}}^{\{p!\}}(\mathbb{R})$ are non-trivial if and only if the weight sequence satisfies the mild lower bound

$$\sup_{p \geq 2} \frac{(\log p)^p}{h^p M_p} < \infty$$

for all $h > 0$ and for some $h > 0$, respectively (cf. Proposition 13.2.8). Finding precise conditions on two weight sequences N_p and M_p that characterize the non-triviality of $\mathcal{S}_{(M_p)}^{(N_p)}(\mathbb{R})$ and $\mathcal{S}_{\{M_p\}}^{\{N_p\}}(\mathbb{R})$ is a long-standing open question, raised by Gelfand and Shilov [63, Chap. 1]; our result then solves this question when one fixes $N_p = p!$.

Our second goal is to give an analytic representation theory for the dual spaces $\mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$. Kawai [87] showed that the space $\mathcal{S}'_{\{p!\}}^{\{p!\}}(\mathbb{R})$ of Fourier hyperfunctions can be represented as the quotient of the space of analytic functions defined outside the real line and having infra-exponential growth outside every strip containing the real line modulo its subspace of entire functions of infra-exponential type. Similarly, Silva [152] showed that the space $\mathcal{S}'_{(p!)}^{(p!)}(\mathbb{R})$ of Fourier ultrahyperfunctions can be represented as the quotient of the space of analytic functions defined outside some strip and having exponential growth modulo its subspace of entire functions of exponential type; in fact, in the cohomological approach of Kawai and Silva, these spaces are initially defined as such. We will extend these results to (ultra)hyperfunctions of fast growth. More precisely, we show that every ultrahyperfunction of type (ω) (hyperfunction of type $\{\omega\}$, respectively) can be represented as the boundary value of an analytic function defined outside some strip

(outside the real line, respectively) and satisfying bounds of the type $O(e^{\omega(\lambda|\operatorname{Re} z|)})$ for some $\lambda > 0$ (for every $\lambda > 0$ and outside every strip containing the real line, respectively). Furthermore, we prove a result concerning the analytic continuation of functions whose boundary value give rise to the zero functional, which can either be viewed as a weighted version of Painlevé's theorem on analytic continuation or as a one-dimensional version of the edge of the wedge theorem. These two types of results will enable us to express $\mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$ as quotients of certain weighted spaces of analytic functions.

Finally, we develop a local theory of (ultra)hyperfunctions of fast growth. In [152], Silva introduced a useful notion of real support for ultrahyperfunctions. We extend such considerations to (ultra)hyperfunctions of fast growth and prove a support splitting theorem similar to Proposition 2.2.14. Based upon this result, we shall construct analytic representations of ultradistributions of exponential type via the Laplace transform of (ultra)hyperfunctions of fast growth in Chapter 14.

This chapter is organized as follows. In Section 13.2, we show some useful properties of weight functions. Many crucial arguments in the sequel depend upon the existence of analytic functions satisfying certain lower and upper bounds with respect to a weight function on a strip. Section 13.3 is devoted to the construction of such analytic functions. We also prove there a quantified Phragmén-Lindelöf type result for analytic functions defined on strips. Basic properties of the test function spaces $\mathcal{U}_{(\omega)}(\mathbb{C})$ and $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ are discussed in Section 13.4. In particular, we show the non-triviality result for $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ and determine their images under the Fourier transform. In Section 13.5, we present the analytic representation theory for $\mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$, and, as an application, we characterize the non-triviality of the space $\mathcal{U}_{(\omega)}(\mathbb{C})$. We introduce the

notion of (real) support in Section 13.6 and provide a support splitting theorem there. Finally, in Section 13.7, we give a variant of the theory from the previous sections that applies to spaces defined via subadditive weights. For a weight function ω , the modification consists in replacing (12.0.1) in the definition of the test function spaces by estimates of the form

$$\sup_{z \in T^h} |\varphi(z)| e^{\lambda \omega(|\operatorname{Re} z|)} < \infty.$$

These spaces will play a crucial role in Chapter 14.

13.2 Weight functions

In this preliminary section, we prove some auxiliary results on weight functions that will be used later on in this chapter. We also discuss the particular case when the weight function is given by the associated function of a weight sequence.

A *weight function* is simply an increasing function $\omega : [0, \infty) \rightarrow [0, \infty)$. Unless explicitly stated otherwise, we shall always assume throughout this chapter that ω satisfies

$$\lim_{t \rightarrow \infty} \frac{\omega(t)}{\log t} = \infty. \quad (13.2.1)$$

We shall make use of some of the following conditions:

$$(\delta) \quad 2\omega(t) \leq \omega(Ht) + \log C_0, \quad t \geq 0, \text{ for some } C_0, H \geq 1.$$

$$(\epsilon)_0 \quad \int_0^\infty \omega(t) e^{-\mu t} dt < \infty \text{ for all } \mu > 0.$$

$$(\epsilon)_\infty \quad \int_0^\infty \omega(t) e^{-\mu t} dt < \infty \text{ for some } \mu > 0.$$

We also introduce the following quantified version of $(\epsilon)_0$ and $(\epsilon)_\infty$:

$$(\epsilon)_\mu \quad \int_0^\infty \omega(t) e^{-\mu t} dt < \infty, \quad \mu > 0.$$

Remark 13.2.1. In the sequel, the symbols C_0 and H will *always* refer to the constants appearing in (δ) .

The function ω is extended to the whole real line via $\omega(t) = \omega(|t|)$, $t \in \mathbb{R}$. Furthermore, we employ the short-hand notation $\omega_\lambda = \omega(\lambda \cdot)$ for $\lambda > 0$. The relation $\omega \subset \sigma$ between two weight functions means that there are $C, \lambda > 0$ such that

$$\sigma(t) \leq \omega_\lambda(t) + C, \quad t \geq 0.$$

The stronger relation $\omega \prec \sigma$ means that the latter inequality remains valid for every $\lambda > 0$ and suitable $C = C_\lambda > 0$. The reader should keep in mind that these two relations “reverse” orders of growth. The weight functions ω and σ are said to be equivalent, denoted by $\omega \sim \sigma$, if both $\omega \subset \sigma$ and $\sigma \subset \omega$ hold.

Example 13.2.2.

- $t^s, \quad s > 0.$
- $\exp(t^s \log^r(1+t)), \quad 0 \leq s < 1, r \geq 0, s+r > 0.$
- $\exp\left(\frac{t}{\log^s(e+t)}\right), \quad s > 0.$
- $e^t.$

All these weight functions satisfy (δ) . Moreover, the first three of them fulfill $(\epsilon)_0$ while the last one satisfies $(\epsilon)_\infty$ but not $(\epsilon)_0$.

The next lemma gives a pointwise characterization of the conditions $(\epsilon)_0$ and $(\epsilon)_\infty$.

Lemma 13.2.3. *Let $\nu > \mu > 0$ and suppose that ω is a weight function satisfying $(\epsilon)_\mu$, then $\omega(t) = o(e^{\nu t})$. Consequently, ω satisfies $(\epsilon)_0$ ($(\epsilon)_\infty$, respectively) if and only if $e^t \prec \omega$ ($e^t \subset \omega$, respectively).*

Proof. Suppose the opposite, then there would exist $\varepsilon > 0$ and a sequence of positive numbers $(t_n)_n$ such that $\omega(t_n) \geq \varepsilon e^{\nu t_n}$ and $t_{n+1} \geq \nu t_n / \mu$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} \int_0^\infty \omega(t) e^{-\mu t} dt &\geq \sum_{n=0}^\infty \int_{t_n}^{t_{n+1}} \omega(t) e^{-\mu t} dt \\ &\geq \left(\frac{\nu}{\mu} - 1 \right) t_0 \sum_{n=0}^\infty \omega(t_n) e^{-\nu t_n}. \end{aligned}$$

Since the last series is divergent, this contradicts $(\epsilon)_\mu$. □

We now show three useful lemmas.

Lemma 13.2.4. *Let ω be a weight function satisfying (δ) and $(\epsilon)_0$ $((\epsilon)_\infty, \text{ respectively})$. Then, there is another weight function σ with $\omega \sim \sigma$ that satisfies (δ) , $(\epsilon)_0$ $((\epsilon)_\infty, \text{ respectively})$, and the additional condition*

$$(\zeta) \quad \lim_{t \rightarrow \infty} \sigma(\lambda t) - \sigma(t) = \infty, \quad \forall \lambda > 1.$$

Proof. Let $(t_n)_n$ be an increasing sequence of non-negative numbers with $t_0 = 0$ and $t_n \rightarrow \infty$ such that $\omega(t) \geq n \log t$ for all $t \geq t_n$. Define $\rho(t) = n \log t$ for $t \in [t_n, t_{n+1})$ and $\sigma = \omega + \rho$. Observe that $\omega(t) \leq \sigma(t) \leq 2\omega(t)$ for all $t \geq 0$. Hence, condition (δ) implies that ω and σ are equivalent weight functions. Since (δ) and $(\epsilon)_0$ $((\epsilon)_\infty, \text{ respectively})$ are invariant under the relation \sim , the weight σ satisfies these conditions as well. For $\lambda > 1$ and $t \in [t_n, t_{n+1})$, we have that

$$\sigma(\lambda t) - \sigma(t) \geq \rho(\lambda t) - \rho(t) \geq n(\log(\lambda t) - \log t) = n \log \lambda,$$

whence (ζ) follows. □

Lemma 13.2.5. *Let ω be a weight function satisfying $(\epsilon)_0$ $((\epsilon)_\infty, \text{ respectively})$. Then, there is a weight function σ satisfying (δ) , $(\epsilon)_0$ $((\epsilon)_\infty, \text{ respectively})$ and $\omega(t) \leq \sigma(t)$ for all $t \geq 0$.*

Proof. Set $\sigma(t) = \int_0^{t+1} \omega(x)dx$. The condition $(\epsilon)_0$ $((\epsilon)_\infty$, respectively) clearly holds for σ . We also have $\sigma(t) \geq \int_t^{t+1} \omega(x)dx \geq \omega(t)$ for all $t \geq 0$. Finally, since ω is increasing, we obtain that

$$2\sigma(t) \leq \int_0^{t+1} \omega(x)dx + \int_0^{t+1} \omega(x+t+1)dx = \int_0^{2t+2} \omega(x)dx \leq \sigma(3t),$$

for $t \geq 1$, whence (δ) follows. \square

Lemma 13.2.6. *Let ω be a weight function satisfying $(\epsilon)_0$. Then, there is a weight function σ satisfying $(\epsilon)_0$ such that $\omega_\lambda(t) = o(\sigma(t))$ for all $\lambda > 0$.*

Proof. We inductively determine a sequence of non-negative numbers $(t_n)_{n \geq 1}$ with $t_1 = 0$ that satisfies

$$\int_{t_n}^{\infty} \omega(t)e^{-t/n^2} dt \leq \frac{1}{2^n}, \quad \frac{t_n}{n} \geq \frac{t_{n-1}}{n-1} + 1, \quad n \geq 2.$$

Define $\sigma(t) = n\omega(nt)$ for $t \in [t_n/n, t_{n+1}/(n+1))$. Clearly, σ is a weight function and $\omega_\lambda(t) = o(\sigma(t))$ for all $\lambda > 0$. Moreover, for all $n_0 \geq 1$ it holds that

$$\begin{aligned} \int_0^{\infty} \sigma(t)e^{-t/n_0} dt &\leq \int_0^{t_{n_0}/n_0} \sigma(t)e^{-t/n_0} dt \\ &\quad + \sum_{n=n_0}^{\infty} \int_{t_n/n}^{t_{n+1}/(n+1)} n\omega(nt)e^{-t/n} dt \\ &\leq t_{n_0}\omega(t_{n_0}) + \sum_{n=n_0}^{\infty} \int_{t_n}^{\infty} \omega(t)e^{-t/n^2} dt \\ &\leq t_{n_0}\omega(t_{n_0}) + \sum_{n=n_0}^{\infty} \frac{1}{2^n} < \infty. \end{aligned}$$

\square

Finally, we consider the case when the weight function is given by the associated function of a weight sequence M_p . To this end, we introduce the ensuing two new conditions on weight sequences:

$$(M.5)_0 \quad \sum_{p=1}^{\infty} e^{-\mu m_p} < \infty \text{ for all } \mu > 0.$$

$$(M.5)_{\infty} \quad \sum_{p=1}^{\infty} e^{-\mu m_p} < \infty \text{ for some } \mu > 0.$$

Let N_p be another weight sequence and denote by M and N the associated functions of M_p and N_p , respectively. If M_p and N_p both satisfy (M.1), then $M_p \subset N_p$ ($M_p \prec N_p$, respectively) if and only if $M \subset N$ ($M \prec N$, respectively) (cf. Subsection 2.2.1) and therefore our use of the symbols \subset and \prec for weight functions is consistent with that for weight sequences.

Example 13.2.7.

- $p!^s, \quad s > 0.$
- $\log(p+e)^{s(p+e)^r}, \quad s, r \geq 1, sr > 1.$
- $\log(p+e)^{(p+e)}.$

The first two of these weight sequences fulfil $(M.5)_0$ while the last one satisfies $(M.5)_{\infty}$ but not $(M.5)_0$.

The next proposition characterizes $(\epsilon)_0$ and $(\epsilon)_{\infty}$ for M in terms of the weight sequence M_p itself. As before, we denote by m the counting function of the sequence $(m_p)_{p \geq 1}$, that is,

$$m(t) = \sum_{m_p \leq t} 1, \quad t \geq 0.$$

If M_p satisfies (M.1), the function M can be represented as follows [89, Equation (3.11), p. 50]

$$M(t) = \int_0^t \frac{m(\lambda)}{\lambda} d\lambda, \quad t \geq 0. \quad (13.2.2)$$

Proposition 13.2.8. *Let M_p be a weight sequence satisfying (M.1). Then, M satisfies (δ) if and only if M_p satisfies (M.2). Moreover, the following statements are equivalent:*

- (i) M satisfies $(\epsilon)_0$ $((\epsilon)_\infty$, respectively).
- (ii) m satisfies $(\epsilon)_0$ $((\epsilon)_\infty$, respectively).
- (iii) $e^t \prec M$ ($e^t \subset M$, respectively).
- (iv) $e^t \prec m$ ($e^t \subset m$, respectively).
- (v) M_p satisfies $(M.5)_0$ $((M.5)_\infty$, respectively).
- (vi) $\log(p+e)^{(p+e)} \prec M_p$ ($\log(p+e)^{(p+e)} \subset M_p$, respectively).

Proof. For the equivalence between (M.2) and (δ) we refer to Subsection 2.2.1. Next, integration by parts yields that

$$\sum_{m_p \leq t} e^{-\mu m_p} = \int_0^t e^{-\mu \lambda} dm(\lambda) = m(t)e^{-\mu t} + \mu \int_0^t m(\lambda)e^{-\mu \lambda} d\lambda$$

for all $t \geq 0$. Moreover, by (13.2.2), we obtain that

$$\int_0^t M(\lambda)e^{-\mu \lambda} d\lambda = \frac{1}{\mu} \int_0^t \frac{m(\lambda)e^{-\mu \lambda}}{\lambda} d\lambda - \frac{M(t)e^{-\mu t}}{\mu}.$$

Lemma 13.2.3 now implies that (i)-(v) are equivalent to one another. Since the associated function of the sequence $\log(p+e)^{(p+e)}$ is equivalent to e^t , conditions (iii) and (vi) are also equivalent to each other. \square

13.3 Analytic functions in a strip

The goal of this section is to construct functions that are analytic and satisfy certain lower and upper bounds with respect to a weight function in a given horizontal strip of the complex plane. Since the functions we aim to construct are zero-free, we first study harmonic functions in a strip. We also show a Phragmén-Lindelöf type result for analytic functions defined in strips and having decay with respect to a weight function; Proposition 13.3.5 actually delivers a

useful three lines type inequality. As usual, $\mathcal{O}(\Omega)$ stands for the space of analytic functions in an open set $\Omega \subseteq \mathbb{C}$. For $h > 0$ we set $T^h = \mathbb{R} + i(-h, h)$, $T_+^h = \mathbb{R} + i(0, h)$, and $T_-^h = \mathbb{R} + i(-h, 0)$. Furthermore, we write $z = x + iy \in \mathbb{C}$ for a complex variable.

The Poisson kernel of the strip T_+^π is given by

$$P(x, y) = \frac{\sin y}{\cosh x - \cos y}$$

and has the ensuing properties [165]:

- $P(x, y)$ is harmonic in $T_+^{2\pi}$.
- $P(x, y) > 0$ in T_+^π .
- $|P(x, y)| \leq \frac{|\sin y|e^{-|x|+1}}{\cosh 1 - 1}$, $|x| \geq 1$, $0 < y < 2\pi$.
- $\int_0^\infty P(x, y)dx = \pi - y$, $0 < y < \pi$.

We employ the short-hand notation

$$P_h(x, y) = P\left(\frac{\pi x}{h}, \frac{\pi y}{h}\right), \quad x + iy \in T_+^h.$$

The Poisson transform with respect to the strip T_+^h of a measurable function f in \mathbb{R} is defined as

$$P_h\{f; x, y\} := \frac{1}{2h} \int_{-\infty}^\infty P_h(t - x, y)f(t)dt.$$

Lemma 13.3.1. *Let ω be a weight function satisfying $(\epsilon)_{\pi/h}$. Then, its Poisson transform is harmonic in T_+^h and satisfies the lower bound*

$$P_h\{\omega; x, y\} \geq \frac{\omega(x)}{2} \left(1 - \frac{y}{h}\right), \quad x + iy \in T_+^h.$$

Moreover, if ω satisfies $(\epsilon)_{\pi/(2h)}$, then the upper bound

$$P_h\{\omega; x, y\} \leq \left(\omega(2x) + \omega\left(\frac{2h}{\pi}\right)\right) \left(1 - \frac{y}{h}\right) + C, \quad x + iy \in T_+^h,$$

holds as well, where

$$C = \frac{e}{2h(\cosh 1 - 1)} \int_0^\infty e^{-\pi t/(2h)} \omega(t) dt < \infty.$$

Proof. Since

$$P_h\{\omega; x, y\} = P_\pi\left\{\omega_{h/\pi}; \frac{\pi x}{h}, \frac{\pi y}{h}\right\},$$

we may assume that $h = \pi$. Set $P\{\omega; x, y\} = P_\pi\{\omega; x, y\}$. The function $P\{\omega; x, y\}$ is harmonic in T_+^π because ω satisfies $(\epsilon)_1$ [165, Thm. 1]. By the symmetry properties of the weight function ω and the Poisson kernel P , it suffices to show the inequalities for $x \geq 0$. We have that

$$\begin{aligned} P\{\omega; x, y\} &= \frac{1}{2\pi} \int_{-\infty}^\infty P(t, y) \omega(t+x) dt \geq \frac{1}{2\pi} \int_0^\infty P(t, y) \omega(t+x) dt \\ &\geq \frac{\omega(x)}{2\pi} \int_0^\infty P(t, y) dt = \frac{\omega(x)}{2} \left(1 - \frac{y}{\pi}\right). \end{aligned}$$

Next, assume that ω satisfies $(\epsilon)_{1/2}$. Then,

$$\begin{aligned} P\{\omega; x, y\} &\leq \frac{1}{\pi} \int_0^\infty P(t, y) \omega(t+x) dt \\ &= \frac{1}{\pi} \int_0^x P(t, y) \omega(t+x) dt + \frac{1}{\pi} \int_x^\infty P(t, y) \omega(t+x) dt \\ &\leq \frac{\omega(2x)}{\pi} \int_0^\infty P(t, y) dt + \frac{1}{\pi} \int_0^\infty P(t, y) \omega(2t) dt \\ &\leq \omega(2x) \left(1 - \frac{y}{\pi}\right) + \frac{1}{\pi} \int_0^\infty P(t, y) \omega(2t) dt. \end{aligned}$$

The result now follows from the fact that

$$\begin{aligned} &\int_0^\infty P(t, y) \omega(2t) dt \\ &= \int_0^1 P(t, y) \omega(2t) dt + \int_1^\infty P(t, y) \omega(2t) dt \\ &\leq \omega(2)(\pi - y) + \frac{e}{2(\cosh 1 - 1)} \int_0^\infty e^{-t/2} \omega(t) dt < \infty. \end{aligned}$$

□

Lemma 13.3.1 has the following important consequence.

Proposition 13.3.2. *Let $h, \lambda > 0$ and let ω be a weight function satisfying $(\epsilon)_{\pi/(8h\lambda)}$. Then, there is $F \in \mathcal{O}(T^h)$ such that*

$$e^{\omega(\lambda x)} \leq |F(z)| \leq Ce^{4\omega(2\lambda x)}, \quad z \in T^h,$$

for some $C = C_{h,\lambda} > 0$. If, in addition, ω satisfies (δ) , then

$$|F(z)| \leq C_0^3 Ce^{\omega(2H^2\lambda x)}, \quad z \in T^h.$$

Proof. Define $U(x, y) = 4P_{4h}\{\omega_\lambda; x, y + h\}$. Lemma 13.3.1 implies that $F(z) = e^{U(x,y)+iV(x,y)}$, with V the harmonic conjugate of U , satisfies all requirements. \square

Remark 13.3.3. Let ω be a weight function (not necessarily satisfying (13.2.1)) that is subadditive, i.e. $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$ for all $t_1, t_2 \geq 0$. Observe that subadditivity implies that $\omega(t) = O(t)$; in particular, $(\epsilon)_0$ holds. As in the proof of Lemma 13.3.1, one can show that the Poisson transform with respect to the strip T_+^h of ω is harmonic and satisfies

$$\frac{\omega(x)}{2} \left(1 - \frac{y}{h}\right) \leq P_h\{\omega; x, y\} \leq \left(\omega(x) + \omega\left(\frac{h}{\pi}\right)\right) \left(1 - \frac{y}{h}\right) + C$$

for $x + iy \in T_+^h$, where

$$C = \frac{e}{h(\cosh 1 - 1)} \int_0^\infty e^{-\pi t/h} \omega(t) dt < \infty.$$

Hence, for all $\lambda, h > 0$, there is $F \in \mathcal{O}(T^h)$ such that

$$e^{\lambda\omega(x)} \leq |F(z)| \leq Ce^{4\lambda\omega(x)}, \quad z \in T^h,$$

for some $C = C_{h,\lambda} > 0$. Subadditive weight functions will play an important role in Chapter 14.

We end this section with a Phragmén-Lindelöf type result for analytic functions defined in strips. We need the following lemma.

Lemma 13.3.4. *Let φ be analytic and bounded in the strip T_+^h and continuous in $\overline{T_+^h}$. If φ is not identically zero, then*

- (i) $-\infty < \int_{-\infty}^{\infty} \log |\varphi(x)| e^{-\pi|x|/h} dx$ and $-\infty < \int_{-\infty}^{\infty} \log |\varphi(x + ih)| e^{-\pi|x|/h} dx$.
- (ii) $\log |\varphi(z)| \leq P_h\{\log |\varphi|; x, y\} + P_h\{\log |\varphi(\cdot + ih)|; x, h - y\}$ for all $z \in T_+^h$.

Proof. We may assume that $h = \pi$. By employing the conformal mapping

$$z \rightarrow \frac{i - e^z}{i + e^z}$$

from the strip T_+^π onto the unit disk, the results can be derived from some well known results on subharmonic functions in the unit disk, see e.g. [143, Chap. 11] or [34, Chap. X]. \square

Proposition 13.3.5. *Let ω be a weight function satisfying $(\epsilon)_{\pi/h}$. Let φ be holomorphic in the strip T_+^h and continuous in $\overline{T_+^h}$. Suppose that there are $M, C > 0$ such that $|\varphi(z)| \leq M$ for all $z \in T_+^h$ and $|\varphi(x)| \leq Ce^{-\omega(x)}$ for all $x \in \mathbb{R}$. Then,*

$$|\varphi(z)| \leq M^{y/h} C^{1-(y/h)} \exp\left(-\frac{\omega(x)}{2} \left(1 - \frac{y}{h}\right)\right)$$

for all $z = x + iy \in T_+^h$.

Proof. We may assume that φ is not identically zero. By Lemma 13.3.4(ii) we have that

$$\begin{aligned} & |\varphi(z)| e^{P_h\{\omega; x, y\}} \\ & \leq M^{y/h} \exp\left(\frac{1}{2h} \int_{-\infty}^{\infty} (\log |\varphi(t)| + \omega(t)) P_h(t - x, y) dt\right). \end{aligned}$$

By applying Jensen's inequality to the unit measure

$$\frac{P_h(t - x, y)}{(2(h - y))} dt,$$

with $x \in \mathbb{R}$ and $0 < y < h$ fixed, we obtain that

$$\begin{aligned} & |\varphi(z)| e^{P_h\{\omega; x, y\}} \\ & \leq \frac{M^{y/h}}{2(h-y)} \int_{-\infty}^{\infty} \exp\left(\left(1 - \frac{y}{h}\right) (\log |\varphi(t)| + \omega(t))\right) P_h(t-x, y) dt \\ & \leq M^{y/h} C^{1-(y/h)}. \end{aligned}$$

The result now follows from Lemma 13.3.1(i). \square

13.4 Weighted spaces of analytic functions

We now discuss some basic properties of the spaces $\mathcal{U}_{(\omega)}(\mathbb{C})$ and $\mathcal{A}_{\{\omega\}}(\mathbb{R})$. More precisely, we characterize the non-triviality of the space $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ in terms of the growth order of ω and determine the images of these test function spaces under the Fourier transform, thereby extending various results from [27]. Throughout the rest of this chapter the parameters h , k , λ , b , and R always stand for positive real numbers.

Let ω be a weight function. We write \mathcal{A}_{ω}^h for the Banach space consisting of all $\varphi \in \mathcal{O}(T^h)$ satisfying

$$\|\varphi\|_{\mathcal{A}_{\omega}^h} := \sup_{z \in T^h} |\varphi(z)| e^{\omega(x)} < \infty.$$

We set $\mathcal{A}_{\omega_{\lambda}}^h = \mathcal{A}_{\omega}^{h, \lambda}$ and $\|\cdot\|_{\mathcal{A}_{\omega_{\lambda}}^h} = \|\cdot\|_{\mathcal{A}_{\omega}^{h, \lambda}}$.

Lemma 13.4.1. *Let ω and σ be two weight functions such that*

$$\lim_{t \rightarrow \infty} \sigma(t) - \omega(t) = \infty. \quad (13.4.1)$$

Then, for $0 < h < k$, the restriction mapping $\mathcal{A}_{\sigma}^k \rightarrow \mathcal{A}_{\omega}^h$ is injective and compact.

Proof. The compactness follows from Montel's theorem while the injectivity is a consequence of the uniqueness property of analytic functions. \square

We define

$$\mathcal{U}_{(\omega)}(\mathbb{C}) := \varprojlim_{h \rightarrow \infty} \mathcal{A}_{\omega}^{h,h}, \quad \mathcal{A}_{\{\omega\}}(\mathbb{R}) := \varinjlim_{h \rightarrow 0^+} \mathcal{A}_{\omega}^{h,h}, \quad (13.4.2)$$

and

$$\mathcal{A}_{(\omega)}^h := \varprojlim_{k \rightarrow h^-} \varprojlim_{\lambda \rightarrow \infty} \mathcal{A}_{\omega}^{k,\lambda}, \quad \mathcal{A}_{\{\omega\}}^h := \varinjlim_{k \rightarrow h^+} \varinjlim_{\lambda \rightarrow 0^+} \mathcal{A}_{\omega}^{k,\lambda}. \quad (13.4.3)$$

If ω satisfies (δ) , then $\omega_H(t) - \omega(t) \rightarrow \infty$, whence Lemma 13.4.1 implies that $\mathcal{U}_{(\omega)}(\mathbb{C})$ and $\mathcal{A}_{(\omega)}^h$ are (FS) -spaces while $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ and $\mathcal{A}_{\{\omega\}}^h$ are (DFS) -spaces. Let σ be another weight function such that $\omega \sim \sigma$, then $\mathcal{U}_{(\omega)}(\mathbb{C}) = \mathcal{U}_{(\sigma)}(\mathbb{C})$ and $\mathcal{A}_{\{\omega\}}(\mathbb{R}) = \mathcal{A}_{\{\sigma\}}(\mathbb{R})$. The same is true for the spaces (13.4.3). The elements of the dual spaces $\mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$ are called *ultrahyperfunctions of type (ω)* and *hyperfunctions of type $\{\omega\}$* , respectively. As already mentioned, $\mathcal{U}_{(t)}(\mathbb{C})$ and $\mathcal{A}_{\{t\}}(\mathbb{R})$ are the test function spaces for the Fourier ultrahyperfunctions and the Fourier hyperfunctions, respectively. More generally, if $\omega = M$ is the associated function of a weight sequence M_p , then $\mathcal{U}_{(M)}(\mathbb{C}) = \mathcal{S}_{(M_p)}^{(p!)}(\mathbb{R})$ and $\mathcal{A}_{\{M\}}(\mathbb{R}) = \mathcal{S}_{\{M_p\}}^{\{p!\}}(\mathbb{R})$ (cf. Section 9.2).

It is a priori not clear whether the spaces (13.4.2) and (13.4.3) should contain functions that are not identically zero. In the first part of this section, we address the non-triviality of the spaces (13.4.3), and thus, in particular, that of $\mathcal{A}_{\{\omega\}}(\mathbb{R}) = \bigcup_{h>0} \mathcal{A}_{\{\omega\}}^h$. The analysis of the corresponding problem for $\mathcal{U}_{(\omega)}(\mathbb{C})$ is more demanding and is postponed to the next section. We begin with the following necessary condition for the non-triviality of $\mathcal{A}_{\omega}^{h,\lambda}$.

Proposition 13.4.2. *Let ω be a weight function and suppose that $\mathcal{A}_{\omega}^{h,\lambda}$ contains a function that is not identically zero. Then, ω satisfies $(\epsilon)_{\pi/(h\lambda)}$.*

Proof. Let $\varphi \in \mathcal{A}_{\omega}^{h,\lambda} \setminus \{0\}$. By applying Lemma 13.3.4(i) to $\varphi(\cdot - ik)$, $0 < k < h$, we obtain that

$$\begin{aligned} -\infty &< \int_{-\infty}^{\infty} \log |\varphi(x - ik)| e^{-\pi|x|/h} dx \\ &\leq \log \|\varphi\|_{\mathcal{A}_{\omega}^{h,\lambda}} \int_{-\infty}^{\infty} e^{-\pi|x|/h} dx - \frac{2}{\lambda} \int_0^{\infty} \omega(x) e^{-\pi x/(h\lambda)} dx, \end{aligned}$$

and, thus, $\int_0^{\infty} \omega(x) e^{-\pi x/(h\lambda)} dx < \infty$. \square

The following result is essentially due to Mandelbrojt [108]; see also [77].

Proposition 13.4.3. ([108, Sect. 2.1]) *Let ω be a weight function. Then, the space $\mathcal{A}_{(\omega)}^h$ ($\mathcal{A}_{\{\omega\}}^h$, respectively) is non-trivial if and only if ω satisfies $(\epsilon)_0$ ($(\epsilon)_{\infty}$, respectively). Consequently, $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ is non-trivial if and only if ω satisfies $(\epsilon)_{\infty}$.*

Proof. The direct implication follows from Proposition 13.4.2. If ω satisfies $(\epsilon)_{\infty}$, Proposition 13.3.2 gives the non-triviality of $\mathcal{A}_{\{\omega\}}^h$. Assume now that ω satisfies $(\epsilon)_0$. By Lemma 13.2.6, there is a weight function σ satisfying $(\epsilon)_0$ such that $\omega_{\lambda}(t) = o(\sigma(t))$ for all $\lambda > 0$. By Proposition 13.3.2 there is an analytic function F in T^h such that $|F(z)| \geq e^{\sigma(x)}$ for all $z \in T^h$. Then, $1/F$ is an element of $\mathcal{A}_{(\omega)}^h$ that is not identically zero. \square

The remainder of this section is devoted to computing the images of $\mathcal{U}_{(\omega)}(\mathbb{C})$ and $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ under the Fourier transform. These spaces are the ultradifferentiable counterparts of the space $\mathcal{K}_1(\mathbb{R})$ of exponentially rapidly decreasing smooth functions [75, 83]. Let ω be a weight function. We write $\mathcal{K}_{1,\omega}^h(\mathbb{R})$ for the function space consisting of all $\psi \in L^1(\mathbb{R})$ such that

$$\rho_{\omega}(\psi) := \sup_{x \in \mathbb{R}} |\mathcal{F}^{-1}(\psi)(x)| e^{\omega(x)} < \infty$$

and

$$\rho^h(\psi) := \sup_{\xi \in \mathbb{R}} |\psi(\xi)| e^{h|\xi|} < \infty;$$

it becomes a Banach space when endowed with the norm

$$\|\psi\|_{\mathcal{K}_{1,\omega}^h} := \max\{\rho_\omega(\psi), \rho^h(\psi)\}, \quad \psi \in \mathcal{K}_{1,\omega}^h(\mathbb{R}).$$

We set $\mathcal{K}_{1,\omega_\lambda}^h(\mathbb{R}) = \mathcal{K}_{1,\omega}^{h,\lambda}(\mathbb{R})$ and $\|\cdot\|_{\mathcal{K}_{1,\omega_\lambda}^h} = \|\cdot\|_{\mathcal{K}_{1,\omega}^{h,\lambda}}$. Define

$$\mathcal{K}_{1,(\omega)}(\mathbb{R}) := \varprojlim_{h \rightarrow \infty} \mathcal{K}_{1,\omega}^{h,h}(\mathbb{R}), \quad \mathcal{K}_{1,\{\omega\}}(\mathbb{R}) := \varinjlim_{h \rightarrow 0^+} \mathcal{K}_{1,\omega}^{h,h}(\mathbb{R}).$$

Proposition 13.4.4. *Let ω be a weight function satisfying (δ) and (ϵ_0) ($(\epsilon)_\infty$, respectively). Then, the Fourier transform is a topological isomorphism from $\mathcal{U}_{(\omega)}(\mathbb{C})$ onto $\mathcal{K}_{1,(\omega)}(\mathbb{R})$ (from $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ onto $\mathcal{K}_{1,\{\omega\}}(\mathbb{R})$, respectively).*

Proof. The Fourier transform of an element $\varphi \in \mathcal{A}_\omega^{h,\lambda}$ is given by (cf. [83, p. 167])

$$\widehat{\varphi}(\xi) = \begin{cases} \int_{-\infty}^{\infty} \varphi(x + ik) e^{-i(x+ik)\xi} dx, & \xi \leq 0, \\ \int_{-\infty}^{\infty} \varphi(x - ik) e^{-i(x-ik)\xi} dx, & \xi \geq 0, \end{cases}$$

where $0 < k < h$. This shows that the Fourier transform is a well-defined continuous mapping in both cases. Conversely, let $\psi \in \mathcal{K}_{1,\omega}^{h,\lambda}(\mathbb{R})$. Then, there is $\varphi \in \mathcal{O}(T^h)$ with $\widehat{\varphi} = \psi$ such that

$$|\varphi(z)| \leq \frac{\rho^h(\psi)}{\pi(h-k)}, \quad z \in T^k,$$

where $0 < k < h$. Proposition 13.3.5 and condition (δ) therefore imply that also the inverse Fourier transform is well-defined and continuous in both cases. \square

The elements of $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ and $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$ are called *ultradistributions of class (ω) (of Beurling type) of exponential type* and *ultradistributions of class $\{\omega\}$ (of Roumieu type) of infra-exponential type*, respectively.

Finally, we remark that when $\omega = M$ is the associated function of a weight sequence M_p satisfying (M.1) and (M.2), then $\mathcal{K}_{1,(M)}(\mathbb{R}) = \mathcal{S}_{(p!)}^{(M_p)}(\mathbb{R})$ and $\mathcal{K}_{1,\{M\}}(\mathbb{R}) = \mathcal{S}_{\{p!\}}^{\{M_p\}}(\mathbb{R})$.

13.5 Boundary values

In this section, we develop an analytic representation theory for the spaces $\mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$. Firstly, we show that every ultrahyperfunction of type (ω) (hyperfunction of type $\{\omega\}$, respectively) can be represented as the boundary value of an analytic function defined outside some strip (outside the real line, respectively) and satisfying certain growth bounds with respect to the weight function ω . Silva obtained analytic representations of ultrahyperfunctions via a careful analysis of the Cauchy-Stieltjes transform [152, 83]. We shall follow a similar approach, the functions constructed in Section 13.3 are essential for this method. Furthermore, we present an (ultra)hyperfunctional version of Painlevé's theorem on analytic continuation. This enables us to express the dual spaces $\mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$ as quotients of spaces of analytic functions. Furthermore, we employ these results to characterize the non-triviality of the space $\mathcal{U}_{(\omega)}(\mathbb{C})$.

13.5.1 Two general results on boundary values

For $0 < b < R$ we set $T^{b,R} = T^R \setminus \overline{T^b} = \mathbb{R} + i((-R, -b) \cup (b, R))$. Let ω be a weight function. We define $\mathcal{O}_\omega^{b,R}$ as the Banach space consisting of all $F \in \mathcal{O}(T^{b,R})$ satisfying

$$\|F\|_{\mathcal{O}_\omega^{b,R}} := \sup_{z \in T^{b,R}} |F(z)|e^{-\omega(x)} < \infty$$

and \mathcal{P}_ω^R as the Banach space consisting of all $P \in \mathcal{O}(T^R)$ such that

$$\|P\|_{\mathcal{P}_\omega^R} := \sup_{z \in T^R} |P(z)|e^{-\omega(x)} < \infty.$$

We set $\mathcal{O}_{\omega_\lambda}^{b,R} = \mathcal{O}_\omega^{b,R,\lambda}$, $\mathcal{P}_{\omega_\lambda}^R = \mathcal{P}_\omega^{R,\lambda}$, $\|\cdot\|_{\mathcal{O}_{\omega_\lambda}^{b,R}} = \|\cdot\|_{\mathcal{O}_\omega^{b,R,\lambda}}$, and $\|\cdot\|_{\mathcal{P}_{\omega_\lambda}^R} = \|\cdot\|_{\mathcal{P}_\omega^{R,\lambda}}$. As in Lemma 13.4.1, one easily obtains that:

Lemma 13.5.1. *Let $0 < b < c < L < R$ and let ω and σ be weight functions satisfying (13.4.1). Then, the restriction mappings $\mathcal{O}_\omega^{b,R} \rightarrow \mathcal{O}_\sigma^{c,L}$ and $\mathcal{P}_\omega^R \rightarrow \mathcal{O}_\sigma^L$ are injective and compact.*

Let ω and σ be weight functions satisfying (13.4.1) and

$$\int_0^\infty e^{\omega(t)-\sigma(t)} dt < \infty.$$

If ω satisfies (δ) , the above conditions are fulfilled for $\omega = \omega_\lambda$ and $\sigma = \omega_{H\lambda}$ for each $\lambda > 0$. Let $0 < b < h < R$. Given an analytic function $F \in \mathcal{O}_\omega^{b,R}$, we associate to F an element of $(\mathcal{A}_\sigma^h)'$ via the *boundary value mapping*

$$\langle \text{bv}(F), \varphi \rangle := - \int_{\Gamma_k} F(z) \varphi(z) dz, \quad \varphi \in \mathcal{A}_\sigma^h,$$

where $b < k < h$ and Γ_k is the (counterclockwise oriented) boundary of T^k . By Cauchy's integral theorem, the definition of $\text{bv}(F)$ is independent of the chosen k . The function F is said to be an *analytic representation of f* . We have the following general result on the existence of analytic representations.

Proposition 13.5.2. *Let $0 < k < b < h < R$ and let ω , σ , and κ be three weight functions satisfying*

$$\lim_{t \rightarrow \infty} \sigma(t) - \omega(t) = \infty, \quad \lim_{t \rightarrow \infty} \kappa(t) - \sigma(t) = \infty, \quad (13.5.1)$$

and

$$\int_0^\infty e^{\omega(t)-\sigma(t)} dt < \infty, \quad \int_0^\infty e^{\sigma(t)-\kappa(t)} dt < \infty. \quad (13.5.2)$$

Furthermore, suppose that there is $P \in \mathcal{O}(T^R)$ such that $C_1 e^{\omega(x)} \leq |P(z)| \leq C_2 e^{\sigma(x)}$ for all $z \in T^R$ and some $C_1, C_2 > 0$. Then, every $f \in (\mathcal{A}_\omega^k)'$ is the boundary value of some element of $\mathcal{O}_\sigma^{b,R}$ on \mathcal{A}_κ^h , that is, there is $F \in \mathcal{O}_\sigma^{b,R}$ such that $\text{bv}(F) = f$ on \mathcal{A}_κ^h .

Proof. Cauchy's integral formula yields that

$$\varphi(\zeta) = \frac{1}{2\pi i P(\zeta)} \int_{\Gamma_b} \frac{\varphi(z) P(z)}{z - \zeta} dz, \quad \zeta \in T^k,$$

for each $\varphi \in \mathcal{A}_\kappa^h$. Let $R_n(\zeta)$ be a sequence of Riemann sums converging to the integral in the right-hand side of the above expression. Then, $R_n(\zeta)/(2\pi i P(\zeta)) \rightarrow \varphi(\zeta)$ in \mathcal{A}_ω^k . Hence,

$$\langle f(\zeta), \varphi(\zeta) \rangle = \int_{\Gamma_b} \frac{P(z)}{2\pi i} \left\langle f(\zeta), \frac{1}{(z - \zeta)P(\zeta)} \right\rangle \varphi(z) dz$$

and, thus,

$$F(z) = \frac{P(z)}{2\pi i} \left\langle f(\zeta), \frac{1}{(\zeta - z)P(\zeta)} \right\rangle$$

is an element of $\mathcal{O}_\sigma^{b,R}$ such that $\text{bv}(F) = f$ on \mathcal{A}_κ^h . \square

Our next result shows that functions whose boundary value give rise to the zero functional can be analytically continued.

Proposition 13.5.3. *Let $0 < b < h < R$ and let ω , σ , and κ be three weight functions satisfying (13.5.1) and (13.5.2). Furthermore, suppose that there is $P \in \mathcal{O}(T^R)$ such that $C_1 e^{\sigma(x)} \leq |P(z)| \leq C_2 e^{\kappa(x)}$ for all $z \in T^R$ and some $C_1, C_2 > 0$. If $F \in \mathcal{O}_\omega^{b,R}$ is such that $\text{bv}(F) = 0$ on \mathcal{A}_σ^h , then $F \in \mathcal{P}_\kappa^R$.*

Proof. Let $0 < b < k < h < L < R$. It suffices to show that

$$F(z) = \frac{P(z)}{2\pi i} \int_{\Gamma_L} \frac{F(\zeta)}{(\zeta - z)P(\zeta)} d\zeta, \quad z \in T^{h,L}. \quad (13.5.3)$$

Fix $z \in \mathbb{C}$ with $h < \text{Im } z < L$; the case $-L < \text{Im } z < -h$ is analogous. We denote by Γ^+ (Γ^- , respectively) the part of a contour Γ in the upper (lower, respectively) half-plane. Cauchy's integral formula yields that

$$F(z) = \frac{P(z)}{2\pi i} \left(\int_{\Gamma_L^+} \frac{F(\zeta)}{(\zeta - z)P(\zeta)} d\zeta - \int_{\Gamma_k^+} \frac{F(\zeta)}{(\zeta - z)P(\zeta)} d\zeta \right).$$

Since $1/((\cdot - z)P) \in \mathcal{A}_\sigma^h$, the assumption $\text{bv}(F) = 0$ on \mathcal{A}_σ^h implies that

$$\int_{\Gamma_k^+} \frac{F(\zeta)}{(\zeta - z)P(\zeta)} d\zeta = - \int_{\Gamma_k^-} \frac{F(\zeta)}{(\zeta - z)P(\zeta)} d\zeta.$$

Furthermore, because $1/((\cdot - z)P)$ is analytic in the horizontal strip $-R < \operatorname{Im} \zeta < -b$, Cauchy's integral theorem shows that

$$\int_{\Gamma_k^-} \frac{F(\zeta)}{(\zeta - z)P(\zeta)} d\zeta = \int_{\Gamma_L^-} \frac{F(\zeta)}{(\zeta - z)P(\zeta)} d\zeta,$$

whence (13.5.3) holds. \square

Combining these two results with Proposition 13.3.2, we obtain the following corollaries.

Corollary 13.5.4. *Let $0 < k < b < h < R$ and let ω be a weight function satisfying (δ) and $(\epsilon)_0$. For every $f \in (\mathcal{A}_\omega^{k,\lambda})'$ there is $F \in \mathcal{O}_\omega^{b,R,2H^2\lambda}$ such that $\operatorname{bv}(F) = f$ on $\mathcal{A}_\omega^{h,2H^3\lambda}$.*

Corollary 13.5.5. *Let $0 < b < h < R$ and let ω be a weight function satisfying (δ) and $(\epsilon)_0$. If $F \in \mathcal{O}_\omega^{b,R,\lambda}$ is such that $\operatorname{bv}(F) = 0$ on $\mathcal{A}_\omega^{h,H\lambda}$, then $F \in \mathcal{P}_\omega^{R,2H^3\lambda}$.*

13.5.2 Boundary values of analytic functions in spaces of ultrahyperfunctions of fast growth

We start by studying the dual spaces $(\mathcal{A}_{(\omega)}^h)'$. For $0 < h < R$ we write

$$\mathcal{O}_{(\omega)}^{h,R} := \varinjlim_{b \rightarrow h^-} \varinjlim_{L \rightarrow R^+} \varinjlim_{\lambda \rightarrow \infty} \mathcal{O}_\omega^{b,L,\lambda}, \quad \mathcal{P}_{(\omega)}^R := \varinjlim_{L \rightarrow R^+} \varinjlim_{\lambda \rightarrow \infty} \mathcal{P}_\omega^{L,\lambda}.$$

Lemma 13.5.1 implies that, if ω satisfies (δ) , $\mathcal{O}_{(\omega)}^{h,R}$ and $\mathcal{P}_{(\omega)}^R$ are (DFS) -spaces. Furthermore, the boundary value mapping $\operatorname{bv} : \mathcal{O}_{(\omega)}^{h,R} \rightarrow (\mathcal{A}_{(\omega)}^h)'$ is well-defined and continuous.

Lemma 13.5.6. *Let ω be a weight function satisfying (δ) and $(\epsilon)_0$. For each $\lambda > 0$ the space*

$$\mathcal{A}_{(\omega)}^{h,\infty} := \bigcap_{\mu > 0} \mathcal{A}_\omega^{h,\mu}$$

is dense in $\mathcal{A}_\omega^{h,H\lambda}$ with respect to the norm $\|\cdot\|_{\mathcal{A}_\omega^{h,\lambda}}$.

Proof. Let $\varphi \in \mathcal{A}_{\omega}^{h,H\lambda}$ be arbitrary. Choose $\psi \in \mathcal{A}_{(\omega)}^{h+1}$ with $\psi(0) = 1$ and set $\psi_n = \psi(\cdot/n)$ for $n \geq 1$ (the space $\mathcal{A}_{(\omega)}^{h+1}$ is non-trivial by Proposition 13.4.3). Define $\varphi_n = \varphi\psi_n \in \mathcal{A}_{(\omega)}^{h,\infty}$. Then,

$$\begin{aligned} \|\varphi - \varphi_n\|_{\mathcal{A}_{\omega}^{h,\lambda}} &= \sup_{z \in T^h} |\varphi(z)(1 - \psi(z/n))|e^{\omega(\lambda x)} \\ &\leq C_0 \|\varphi\|_{\mathcal{A}_{\omega}^{h,H\lambda}} \sup_{z \in T^h} |1 - \psi(z/n)|e^{-\omega(\lambda x)}, \end{aligned}$$

which shows the result since $\psi_n \rightarrow 1$ uniformly on compact subsets of T^{h+1} . \square

We are now able to show that $(\mathcal{A}_{(\omega)}^h)'$ is isomorphic to the quotient space $\mathcal{O}_{(\omega)}^{h,R}/\mathcal{P}_{(\omega)}^R$.

Proposition 13.5.7. *Let $0 < h < R$ and let ω be a weight function satisfying (δ) and $(\epsilon)_0$. Then, the following sequence*

$$0 \longrightarrow \mathcal{P}_{(\omega)}^R \longrightarrow \mathcal{O}_{(\omega)}^{h,R} \xrightarrow{\text{bv}} (\mathcal{A}_{(\omega)}^h)' \longrightarrow 0$$

is topologically exact. Moreover, for every $f \in (\mathcal{A}_{(\omega)}^h)'$ one can find $0 < b < h$ and $\lambda > 0$ such that for every $R > h$ there is $F \in \mathcal{O}_{\omega}^{b,R,\lambda}$ such that $\text{bv}(F) = f$.

Proof. By the Pták open mapping theorem it suffices to show that the sequence is algebraically exact. It is clear that $\mathcal{P}_{(\omega)}^R \subseteq \ker \text{bv}$. Conversely, let $F \in \mathcal{O}_{(\omega)}^{h,R}$ and suppose $\text{bv}(F) = 0$ on $\mathcal{A}_{(\omega)}^h$. Let $0 < b < h < R < L$, and $\lambda > 0$ be such that $F \in \mathcal{O}_{\omega}^{b,L,\lambda}$. Since $\mathcal{A}_{(\omega)}^{h,\infty} \subset \mathcal{A}_{(\omega)}^h$, Lebesgue's dominated convergence theorem and Lemma 13.5.6 imply that actually $\text{bv}(F) = 0$ on $\mathcal{A}_{\omega}^{h,H^2\lambda}$. Hence, by Corollary 13.5.5, we have that $F \in \mathcal{P}_{\omega}^{L,2H^4\lambda} \subset \mathcal{P}_{(\omega)}^R$. The second statement (and therefore also the surjectivity of the boundary value mapping) is a consequence of the Hahn-Banach theorem and Corollary 13.5.4. \square

We now proceed with showing that $\mathcal{U}_{(\omega)}(\mathbb{C})$ is non-trivial if and only if ω satisfies $(\epsilon)_0$. To this end, we shall use a well known

result about projective spectra of Fréchet spaces. For later use, we formulate it in terms of general complete metrizable topological spaces. A *projective spectrum \mathcal{X} of topological spaces* is a sequence $(X_N)_N$ of topological spaces together with continuous linking mappings $\varrho_{N+1}^N : X_{N+1} \rightarrow X_N$ for each $N \in \mathbb{N}$. The projective limit $\text{proj}^0 \mathcal{X}$ and the notion of reducedness is defined as in Subsection 2.3.2. We then have:

Lemma 13.5.8. *Let $\mathcal{X} = (X_N, \varrho_{N+1}^N)_N$ be a projective spectrum of complete metrizable topological spaces. If $\varrho_{N+1}^N : X_{N+1} \rightarrow X_N$ has dense range for each $N \in \mathbb{N}$, then the projective spectrum \mathcal{X} is reduced.*

Theorem 13.5.9. *Let ω be a weight function. Then, the space $\mathcal{U}_{(\omega)}(\mathbb{C})$ is non-trivial if and only if ω satisfies $(\epsilon)_0$.*

Proof. The direct implication follows from Proposition 13.4.2. For the converse, we may assume that ω satisfies (δ) by Lemma 13.2.5. Proposition 13.4.3 ensures that the space $\mathcal{A}_{(\omega)}^h$ is non-trivial for each $h > 0$ and, thus, by Lemma 13.5.8, it suffices to show that $\mathcal{A}_{(\omega)}^h$ is dense in $\mathcal{A}_{(\omega)}^k$ for all $0 < k < h$. This is a consequence of the Hahn-Banach theorem and Proposition 13.5.7. \square

In view of Lemma 13.5.8, we have shown the following result during the proof of Theorem 13.5.9.

Corollary 13.5.10. *Let ω be a weight sequence satisfying (δ) and $(\epsilon)_0$. Then, $\mathcal{U}_{(\omega)}(\mathbb{C})$ is dense in $\mathcal{A}_{(\omega)}^h$ for all $h > 0$.*

Combining Lemma 13.5.6 and Corollary 13.5.10, we obtain the ensuing result.

Proposition 13.5.11. *Let ω be a weight sequence satisfying (δ) and $(\epsilon)_0$. Then, we have the dense continuous inclusion $\mathcal{U}_{(\omega)}(\mathbb{C}) \hookrightarrow \mathcal{A}_{\{\omega\}}(\mathbb{R})$.*

By Proposition 13.5.11 we may view $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$ as a subspace of $\mathcal{U}'_{(\omega)}(\mathbb{C})$. Our next goal is to construct global analytic representations of elements of $\mathcal{U}'_{(\omega)}(\mathbb{R})$, that is, analytic representations that are defined everywhere outside some closed horizontal strip. The basic idea is to paste together the analytic representations obtained in Proposition 13.5.7 via a Mittag-Leffler procedure. We define

$$\mathcal{O}_{(\omega)}^h := \bigcup_{\lambda>0} \bigcup_{b<h} \bigcap_{R>b} \mathcal{O}_{\omega}^{b,R,\lambda}, \quad \mathcal{O}_{(\omega)} := \bigcup_{h>0} \mathcal{O}_{(\omega)}^h,$$

and

$$\mathcal{P}_{(\omega)} := \bigcup_{\lambda>0} \bigcap_{R>0} \mathcal{P}_{\omega}^{R,\lambda}.$$

We use the union and intersection notation to emphasize that we do not topologize the latter spaces. The following lemma is needed.

Lemma 13.5.12. *Let $0 < L < R$ and let ω be a weight function satisfying (δ) and $(\epsilon)_0$. Then, $\mathcal{U}_{(\omega)}(\mathbb{C})$ is dense in $\mathcal{P}_{\omega}^{R,\lambda}$ with respect to the norm $\|\cdot\|_{\mathcal{P}_{\omega}^{L,H\lambda}}$.*

Proof. By Corollary 13.5.10 it suffices to show that $\mathcal{A}_{(\omega)}^{R,\infty}$ is dense in $\mathcal{P}_{\omega}^{R,\lambda}$ with respect to the norm $\|\cdot\|_{\mathcal{P}_{\omega}^{L,H\lambda}}$. Let $P \in \mathcal{P}_{\omega}^{R,\lambda}$ be arbitrary. Choose $\varphi \in \mathcal{A}_{(\omega)}^{R,\infty}$ with $\varphi(0) = 1$ and set $\varphi_n = \varphi(\cdot/n)$ for $n \geq 1$. Define $P_n = P\varphi_n \in \mathcal{A}_{(\omega)}^{R,\infty}$. We have that

$$\begin{aligned} \|P - P_n\|_{\mathcal{P}_{\omega}^{L,H\lambda}} &= \sup_{z \in T^L} |P(z)(1 - \varphi(z/n))| e^{-\omega(H\lambda x)} \\ &\leq C_0 \|P\|_{\mathcal{P}_{\omega}^{L,\lambda}} \sup_{z \in T^L} |1 - \varphi(z/n)| e^{-\omega(\lambda x)}, \end{aligned}$$

which shows the result since $\varphi_n \rightarrow 1$ uniformly on compact subsets of T^R . \square

Proposition 13.5.13. *Let ω be a weight function satisfying (δ) and $(\epsilon)_0$. Then, the sequence*

$$0 \longrightarrow \mathcal{P}_{(\omega)} \longrightarrow \mathcal{O}_{(\omega)}^h \xrightarrow{\text{bv}} (\mathcal{A}_{(\omega)}^h)' \longrightarrow 0$$

is exact.

Proof. The equality $\mathcal{P}_{(\omega)} = \ker \text{bv}$ is clear from Proposition 13.5.7. We now show that the boundary value mapping is surjective. Let $f \in (\mathcal{A}_{(\omega)}^h)'$ be arbitrary. By Proposition 13.5.7 there are $0 < b < h$ and $\lambda > 0$ such that for every $n \in \mathbb{N}$ there is $G_n \in \mathcal{O}_{\omega}^{b, b+n+1, \lambda}$ such that $\text{bv}(G_n) = f$. Corollary 13.5.5 and Lemma 13.5.6 yield that $G_{n+1} - G_n = P_n \in \mathcal{P}_{\omega}^{b+n+1, 2H^4\lambda}$. By Lemma 13.5.12 we find $\varphi_n \in \mathcal{U}_{(\omega)}(\mathbb{C})$ such that $\|P_n - \varphi_n\|_{\mathcal{P}_{\omega}^{b+n, 2H^5\lambda}} \leq 2^{-n}$. Define

$$F_n(z) = G_n(z) - \sum_{k=0}^{n-1} \varphi_k + \sum_{k=n}^{\infty} (P_k - \varphi_k), \quad z \in T^{b, b+n}.$$

Then, $F_n \in \mathcal{O}_{\omega}^{b, b+n, 2H^5\lambda}$, $\text{bv}(F_n) = f$, and $F_{n+1}(z) = F_n(z)$ for $z \in T^{b, b+n}$. Define $F(z) = F_n(z)$ for $z \in T^{b, b+n}$. Then, F is a well-defined element of $\mathcal{O}_{(\omega)}^h$ such that $\text{bv}(F) = f$. \square

In particular, we have shown the following representation theorem.

Theorem 13.5.14. *Let ω be a weight function satisfying (δ) and $(\epsilon)_0$. Then, the sequence*

$$0 \longrightarrow \mathcal{P}_{(\omega)} \longrightarrow \mathcal{O}_{(\omega)} \xrightarrow{\text{bv}} \mathcal{U}'_{(\omega)}(\mathbb{C}) \longrightarrow 0$$

is exact.

13.5.3 Boundary values of analytic functions in spaces of hyperfunctions of fast growth

We now turn our attention to the space $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$. In analogy with the previous subsection, we start by studying the dual $(\mathcal{A}_{\{\omega\}}^h)'$. For $0 < h < R$ we set

$$\mathcal{O}_{\{\omega\}}^{h, R, \lambda} := \varprojlim_{b \rightarrow h^+} \varprojlim_{L \rightarrow R^-} \varprojlim_{\mu \rightarrow \lambda^+} \mathcal{O}_{\omega}^{b, L, \mu}, \quad \mathcal{O}_{\{\omega\}}^{h, R} := \varprojlim_{\lambda \rightarrow 0^+} \mathcal{O}_{\{\omega\}}^{h, R, \lambda},$$

and

$$\mathcal{P}_{\{\omega\}}^{R, \lambda} := \varprojlim_{L \rightarrow R^-} \varprojlim_{\mu \rightarrow \lambda^+} \mathcal{P}_{\omega}^{L, \mu}, \quad \mathcal{P}_{\{\omega\}}^R := \varprojlim_{\lambda \rightarrow 0^+} \mathcal{P}_{\{\omega\}}^{R, \lambda}.$$

If ω satisfies (δ) , Lemma 13.5.1 implies that $\mathcal{O}_{\{\omega\}}^{h,R}$ and $\mathcal{P}_{\{\omega\}}^R$ are (FS) -spaces. If ω satisfies condition (ζ) from Lemma 13.2.4, this is also true for $\mathcal{O}_{\{\omega\}}^{h,R,\lambda}$ and $\mathcal{P}_{\{\omega\}}^{R,\lambda}$. Furthermore, the boundary value mapping $\text{bv} : \mathcal{O}_{\{\omega\}}^{h,R} \rightarrow (\mathcal{A}_{\{\omega\}}^h)'$ is well-defined and continuous. We need to the following lemma.

Lemma 13.5.15. *Let $0 < h < R$ and $0 < \lambda < \mu$. Let ω be a weight function satisfying (δ) , $(\epsilon)_0$, and (ζ) . Then, $\mathcal{U}_{(\omega)}(\mathbb{C})$ is dense in $\mathcal{O}_{\{\omega\}}^{h,R,\lambda} \cap \mathcal{P}_{\{\omega\}}^{R,\mu}$ with respect to the topology of $\mathcal{O}_{\{\omega\}}^{h,R,\lambda}$.*

Proof. By Corollary 13.5.10 it is enough to verify that $\mathcal{A}_{(\omega)}^R$ is dense in $\mathcal{O}_{\{\omega\}}^{h,R,\lambda} \cap \mathcal{P}_{\{\omega\}}^{R,\mu}$ with respect to the topology of $\mathcal{O}_{\{\omega\}}^{h,R,\lambda}$. Let $P \in \mathcal{O}_{\{\omega\}}^{h,R,\lambda} \cap \mathcal{P}_{\{\omega\}}^{R,\mu}$ be arbitrary. Choose $\varphi \in \mathcal{A}_{(\omega)}^R$ with $\varphi(0) = 1$ and set $\varphi_n = \varphi(\cdot/n)$ for $n \geq 1$. Define $P_n = P\varphi_n \in \mathcal{A}_{(\omega)}^R$. Let $h < b < L < R$ and $\nu > \lambda$ be arbitrary. For $\lambda < \nu_0 < \nu$ we have that

$$\begin{aligned} \|P - P_n\|_{\mathcal{O}_{\omega}^{b,L,\nu}} &= \sup_{z \in T^{b,L}} |P(z)(1 - \varphi(z/n))| e^{-\omega(\nu x)} \\ &\leq \|P\|_{\mathcal{O}_{\omega}^{b,L,\nu_0}} \sup_{z \in T^{b,L}} |1 - \varphi(z/n)| e^{-(\omega(\nu x) - \omega(\nu_0 x))}, \end{aligned}$$

which shows the result because $\omega(\nu t) - \omega(\nu_0 t) \rightarrow \infty$ and $\varphi_n \rightarrow 1$ uniformly on compact subsets of T^R . \square

Proposition 13.5.16. *Let $0 < h < R$ and let ω be a weight function satisfying (δ) and $(\epsilon)_0$. Then, the sequence*

$$0 \longrightarrow \mathcal{P}_{\{\omega\}}^R \longrightarrow \mathcal{O}_{\{\omega\}}^{h,R} \xrightarrow{\text{bv}} (\mathcal{A}_{\{\omega\}}^h)' \longrightarrow 0$$

is topologically exact.

Proof. By the open mapping theorem it suffices to show that the sequence is algebraically exact. Corollary 13.5.5 implies that $\mathcal{P}_{\{\omega\}}^R = \ker \text{bv}$. We now show that the boundary value mapping is surjective. By Lemma 13.2.4 we may assume that ω satisfies (ζ) . Let $f \in (\mathcal{A}_{\{\omega\}}^h)'$ be arbitrary and define

$$X_N = \{F \in \mathcal{O}_{\{\omega\}}^{h+(1/N),R,1/N} \mid \text{bv}(F) = f \text{ on } \mathcal{U}_{(\omega)}(\mathbb{C})\}, \quad N \geq 1.$$

Corollary 13.5.4 implies that X_n is a non-empty closed subspace of $\mathcal{O}_{\{\omega\}}^{h+(1/N),R,1/N}$ and therefore a complete metrizable topological space (with respect to the relative topology). Consider the projective spectrum $(X_N, \varrho_{N+1}^N)_{N \geq 1}$ with $\varrho_{N+1}^N : X_{N+1} \rightarrow X_N$ the inclusion mappings and denote by X its projective limit. It suffices to show that X is non-empty, which would be implied by Lemma 13.5.8 if we verify that X_{N+1} is dense in X_N . Since, by Corollary 13.5.5, every $F \in X_N$ can be written as $F = G + P$ where $G \in X_{N+1}$ and $P \in \mathcal{O}_{\{\omega\}}^{h+(1/N),R,1/N} \cap \mathcal{P}_{\{\omega\}}^{R,4H^4/N}$, this follows from Lemma 13.5.15. \square

We now construct global analytic representations. Set

$$\mathcal{O}_{\{\omega\}}^h := \varprojlim_{R \rightarrow \infty} \mathcal{O}_{\{\omega\}}^{h,R}, \quad \mathcal{O}_{\{\omega\}} := \varprojlim_{h \rightarrow 0^+} \mathcal{O}_{\{\omega\}}^h, \quad \mathcal{P}_{\{\omega\}} := \varprojlim_{R \rightarrow \infty} \mathcal{P}_{\{\omega\}}^R,$$

which are all (FS) -spaces if ω satisfies (δ) (Lemma 13.5.1).

Proposition 13.5.17. *Let ω be a weight function satisfying (δ) and $(\epsilon)_0$. Then, the sequence*

$$0 \longrightarrow \mathcal{P}_{\{\omega\}} \longrightarrow \mathcal{O}_{\{\omega\}}^h \xrightarrow{\text{bv}} (\mathcal{A}_{\{\omega\}}^h)' \longrightarrow 0$$

is topologically exact.

Proof. By the open mapping theorem it suffices to show that the sequence is algebraically exact. Set $X_N = \mathcal{P}_{\{\omega\}}^{h+N}$ and $Y_N = \mathcal{O}_{\{\omega\}}^{h,h+N}$ for $N \geq 1$. Consider the following short sequence of projective spectra

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & Y_1 & \xrightarrow{\text{bv}} & (\mathcal{A}_{\omega}^h)' \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & X_2 & \longrightarrow & Y_2 & \xrightarrow{\text{bv}} & (\mathcal{A}_{\omega}^h)' \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

By Proposition 13.5.16 this sequence is exact in the category of projective spectra, that is, every horizontal sequence is exact. Moreover, by Lemma 13.5.12, X_{n+1} is dense in X_N for all $N \geq 1$. Hence, Lemma 2.3.6 yields that the sequence

$$0 \longrightarrow \varprojlim_N X_N = \mathcal{P}_{\{\omega\}} \longrightarrow \varprojlim_N Y_N = \mathcal{O}_{\{\omega\}}^h \xrightarrow{\text{bv}} (\mathcal{A}_{\{\omega\}}^h)' \longrightarrow 0$$

is exact. □

We are able to prove the main result of this subsection.

Theorem 13.5.18. *Let ω be a weight function satisfying (δ) and $(\epsilon)_0$. Then, the sequence*

$$0 \longrightarrow \mathcal{P}_{\{\omega\}} \longrightarrow \mathcal{O}_{\{\omega\}} \xrightarrow{\text{bv}} \mathcal{A}'_{\{\omega\}}(\mathbb{R}) \longrightarrow 0$$

is topologically exact.

Proof. By the open mapping theorem it suffices to show that the sequence is algebraically exact. We now set $Y_N = \mathcal{O}_{\{\omega\}}^{1/N}$ and $Z_N = (\mathcal{A}_{\{\omega\}}^{1/N})'$ for $N \geq 1$. Consider the following short sequence of projective spectra

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{P}_{\{\omega\}} & \longrightarrow & Y_1 & \xrightarrow{\text{bv}} & Z_1 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathcal{P}_{\{\omega\}} & \longrightarrow & Y_2 & \xrightarrow{\text{bv}} & Z_2 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

By Proposition 13.5.17 this sequence is exact in the category of projective spectra. Hence, Lemma 2.3.6 yields that the sequence

$$0 \longrightarrow \mathcal{P}_{\{\omega\}} \longrightarrow \varprojlim_N Y_N = \mathcal{O}_{\{\omega\}} \xrightarrow{\text{bv}} \varprojlim_N Z_N = \mathcal{A}'_{\{\omega\}}(\mathbb{R}) \longrightarrow 0$$

is exact. □

13.6 Local theory

Following Silva [152], we introduce a notion of (real) support for (ultra)hyperfunctions of fast growth via the analytic continuation properties of their analytic representations. Most importantly, we will establish a support splitting theorem (cf. Proposition 2.2.14) which states that every (ultra)hyperfunction of fast growth can be written as the sum of two (ultra)hyperfunctions of fast growth having support in the positive and negative half-axis, respectively.

13.6.1 Real support of ultrahyperfunctions of fast growth

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ be the extended real line endowed with its usual topology (two-point compactification of \mathbb{R}). For $0 < b < R$ and $A \subseteq \overline{\mathbb{R}}$ we set $T^R(A) = (A \cap \mathbb{R}) + i(-R, R)$ and $T^{b,R}(A) = T^{b,R} \cup T^R(A)$. Let ω be a weight function. For a proper compact subset K of $\overline{\mathbb{R}}$ with non-empty interior we denote by $\mathcal{O}_\omega^{b,R}(K)$ the Banach space consisting of all $F \in \mathcal{O}(T^{b,R}(\text{int } K))$ satisfying

$$\|F\|_{\mathcal{O}_\omega^{b,R}(K)} := \sup\{|F(z)|e^{-\omega(x)} \mid z \in T^{b,R}(\text{int } K)\} < \infty.$$

For $\Omega \subseteq \overline{\mathbb{R}}$ open we define

$$\mathcal{O}_\omega^{h,R}(\Omega) := \varprojlim_{K \in \Omega} \varprojlim_{b \rightarrow h^+} \varprojlim_{L \rightarrow R^-} \mathcal{O}_\omega^{b,L}(K).$$

Similarly as before, we write $\mathcal{O}_{\omega_\lambda}^{b,R}(K) = \mathcal{O}_\omega^{b,R,\lambda}(K)$, $\mathcal{O}_{\omega_\lambda}^{h,R}(\Omega) = \mathcal{O}_\omega^{h,R,\lambda}(\Omega)$, and $\|\cdot\|_{\mathcal{O}_{\omega_\lambda}^{b,R}(K)} = \|\cdot\|_{\mathcal{O}_\omega^{b,R,\lambda}(K)}$. Furthermore, we set

$$\mathcal{O}_{(\omega)}^{h,R}(\Omega) := \bigcup_{\lambda > 0} \mathcal{O}_\omega^{h,R,\lambda}(\Omega), \quad \mathcal{O}_{(\omega)}(\Omega + i\mathbb{R}) := \bigcup_{\lambda, h > 0} \bigcap_{R > h} \mathcal{O}_\omega^{h,R,\lambda}(\Omega).$$

Suppose that ω satisfies (δ) and $(\epsilon)_0$. Let $f \in \mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\Omega \subseteq \overline{\mathbb{R}}$ be open. We say that f *vanishes in* Ω if there is $F \in \mathcal{O}_{(\omega)}(\Omega + i\mathbb{R})$ such that $\text{bv}(F) = f$. Theorem 13.5.14 implies that in such

a case this property holds for all analytic representations of f in $\mathcal{O}_{(\omega)}$. Moreover, the proof of Proposition 13.5.13 shows that for f to vanish in Ω it suffices that there is $F \in \mathcal{O}_{(\omega)}^{h,R}(\Omega)$, for some $0 < h < R$, such that $\text{bv}(F) = f$.

We define the *real support* of $f \in \mathcal{U}'_{(\omega)}(\mathbb{C})$, denoted by $\text{supp}_{\overline{\mathbb{R}}} f$, as the complement of the largest open subset of $\overline{\mathbb{R}}$ in which f vanishes. For $K \Subset \overline{\mathbb{R}}$ we define

$$\mathcal{U}'_{(\omega)}[K + i\mathbb{R}] := \{f \in \mathcal{U}'_{(\omega)}(\mathbb{C}) \mid \text{supp}_{\overline{\mathbb{R}}} f \subseteq K\},$$

the space of ultrahyperfunctions of type (ω) with real support in K . When I is a closed interval of $\overline{\mathbb{R}}$, $f \in \mathcal{U}'_{(\omega)}[I + i\mathbb{R}]$, and $F \in \mathcal{O}_{(\omega)}^{h,R}(\overline{\mathbb{R}} \setminus I)$ is such that $\text{bv}(F) = f$, Cauchy's integral theorem implies that

$$\langle f, \varphi \rangle = - \int_{\Gamma^b(J)} F(z) \varphi(z) dz, \quad \varphi \in \mathcal{U}_{(\omega)}(\mathbb{C}),$$

where $h < b < R$, J is an interval in $\overline{\mathbb{R}}$ such that $I \Subset J$, and $\Gamma^b(J)$ denotes the boundary of $T^b(J)$. More generally, if $K \Subset \overline{\mathbb{R}}$ and J_1, J_2, \dots, J_n is a finite covering of K by open intervals of the extended real line such that their finite end points do not belong to K , and $f = \text{bv}(F) \in \mathcal{U}'_{(\omega)}[K + i\mathbb{R}]$ with $F \in \mathcal{O}_{(\omega)}^{h,R}(\overline{\mathbb{R}} \setminus K)$, we have the representation

$$\langle f, \varphi \rangle = - \left(\int_{\Gamma^b(J_1)} + \int_{\Gamma^b(J_2)} + \dots + \int_{\Gamma^b(J_n)} \right) F(z) \varphi(z) dz, \quad (13.6.1)$$

for each $\varphi \in \mathcal{U}_{(\omega)}(\mathbb{C})$.

If $K \Subset \mathbb{R}$, the space $\mathcal{U}'_{(\omega)}[K + i\mathbb{R}]$ coincides with the space of analytic functionals. Indeed, for $S \subseteq \mathbb{C}$ closed we set

$$\mathcal{O}[S] = \varinjlim_{S \subset \Omega} \mathcal{O}(\Omega).$$

The Silva-Köthe-Grothendieck theorem [117, Thm. 2.1.3] therefore yields that

$$\mathcal{U}'_{(\omega)}[K + i\mathbb{R}] = \bigcup_{R>0} \mathcal{O}'[K + i[-R, R]] = \mathcal{O}'[K + i\mathbb{R}].$$

Our next goal is to show a support splitting theorem for $\mathcal{U}'_{(\omega)}(\mathbb{C})$; it may be considered as an analogue of Proposition 2.2.14 in the present setting. For $a, b \in \mathbb{R}$ we employ the special notations

$$\mathcal{U}'_{(\omega),a+} = \mathcal{U}'_{(\omega)}[[a, \infty] + i\mathbb{R}], \quad \mathcal{U}'_{(\omega),b-} = \mathcal{U}'_{(\omega)}[[-\infty, b] + i\mathbb{R}].$$

Theorem 13.6.1. *Let $-\infty < a \leq b < \infty$ and let ω be a weight function satisfying (δ) and $(\epsilon)_0$. Then, the sequence*

$$0 \longrightarrow \mathcal{O}'[[a, b] + i\mathbb{R}] \longrightarrow \mathcal{U}'_{(\omega),a+} \times \mathcal{U}'_{(\omega),b-} \xrightarrow{T} \mathcal{U}'_{(\omega)}(\mathbb{C}) \longrightarrow 0$$

is exact, where $T(f_1, f_2) = f_1 - f_2$.

We need to introduce some additional function spaces to prove Theorem 13.6.1. Let $\Omega \subseteq \overline{\mathbb{R}}$ be open and define $\mathcal{A}^h_{\omega}(\Omega)$ as the Banach space consisting of all $\varphi \in \mathcal{O}(T^h(\Omega))$ such that

$$\|\varphi\|_{\mathcal{A}^h_{\omega}(\Omega)} := \sup_{z \in T^h(\Omega)} |\varphi(z)| e^{\omega(x)} < \infty.$$

Set $\mathcal{A}^h_{\omega_{\lambda}}(\Omega) = \mathcal{A}^{h,\lambda}_{\omega}(\Omega)$ and $\|\cdot\|_{\mathcal{A}^h_{\omega_{\lambda}}(\Omega)} = \|\cdot\|_{\mathcal{A}^{h,\lambda}_{\omega}(\Omega)}$. The following refinement of Proposition 13.5.2 holds.

Proposition 13.6.2. *Let $0 < k < b < h < R$ and let U, V, Ω be open subsets of $\overline{\mathbb{R}}$ such that $\overline{U} \Subset V$ and $\overline{V} \Subset \Omega$. Let ω, σ , and κ be three weight functions satisfying (13.5.1) and (13.5.2). Furthermore, suppose that there is $P \in \mathcal{O}(T^R)$ such that $C_1 e^{\omega(x)} \leq |P(z)| \leq C_2 e^{\sigma(x)}$ for all $z \in T^R$ and some $C_1, C_2 > 0$. Then, for every $f \in (\mathcal{A}^k_{\omega}(U))'$ there is $F \in \mathcal{O}^{b,R}_{\sigma}(\overline{\mathbb{R}} \setminus V)$ such that $\text{bv}(F) = f$ on $\mathcal{A}^h_{\kappa}(\Omega)$.*

Proposition 13.3.2 implies the following corollary.

Corollary 13.6.3. *Let $0 < k < b < h < R$ and let U, V, Ω be open subsets of $\overline{\mathbb{R}}$ such that $\overline{U} \Subset V$ and $\overline{V} \Subset \Omega$. Let ω be a weight function satisfying (δ) and $(\epsilon)_0$. Then, for every $f \in (\mathcal{A}^{k,\lambda}_{\omega}(U))'$ there is $F \in \mathcal{O}^{b,R,2H^2\lambda}_{\omega}(\overline{\mathbb{R}} \setminus V)$ such that $\text{bv}(F) = f$ on $\mathcal{A}^{h,2H^3\lambda}_{\omega}(\Omega)$.*

Proof of Theorem 13.6.1. Theorem 13.5.14 implies that $\mathcal{O}'[[a, b] + i\mathbb{R}] = \mathcal{U}'_{(\omega), a+} \cap \mathcal{U}'_{(\omega), b-}$. It remains to show that T is surjective. By Lemma 13.2.4 we may assume that ω satisfies (ζ) . Let $h, \lambda > 0$ be arbitrary. Define

$$X^{h, \lambda} := \varinjlim_{\mu \rightarrow \lambda^+} \varinjlim_{k \rightarrow h^+} \mathcal{A}_\omega^{k, \mu},$$

$$X_{a+}^{h, \lambda} := \varinjlim_{\mu \rightarrow \lambda^+} \varinjlim_{k \rightarrow h^+} \varinjlim_{\varepsilon \rightarrow 0^+} \mathcal{A}_\omega^{k, \mu}((a - \varepsilon, \infty]),$$

and

$$X_{b-}^{h, \lambda} := \varinjlim_{\mu \rightarrow \lambda^+} \varinjlim_{k \rightarrow h^+} \varinjlim_{\varepsilon \rightarrow 0^+} \mathcal{A}_\omega^{k, \mu}([-\infty, b + \varepsilon)).$$

Condition (ζ) implies that $X^{h, \lambda}$, $X_{a+}^{h, \lambda}$, and $X_{b-}^{h, \lambda}$ are (DFS) -spaces. Let $g \in (X_{a+}^{h, \lambda})'$ be arbitrary and choose $R > h$. By Corollary 13.6.3, we have that, for every $\varepsilon > 0$ and $k > h$, there is $G_{\varepsilon, k} = G \in \mathcal{O}_\omega^{k, R, 4H^2\lambda}([-\infty, a - \varepsilon])$ such that $\text{bv}(G) = g$. By a similar Mittag-Leffler procedure as in the proof of Proposition 13.5.13, one can now show that there is $G \in \mathcal{O}_\omega^{h, R, 8H^7\lambda}([-\infty, a))$ such that $\text{bv}(G) = g$. Likewise, it holds that for every $g \in (X_{b-}^{h, \lambda})'$ there is $G \in \mathcal{O}_\omega^{h, R, 8H^7\lambda}((b, \infty])$ such that $\text{bv}(G) = g$. Since every element $f \in \mathcal{U}'_{(\omega)}(\mathbb{C})$ can be extended to a continuous linear functional on $X^{h, \lambda}$ for some $h, \lambda > 0$, it suffices to show that the mapping

$$(X_{a+}^{h, \lambda})' \times (X_{b-}^{h, \lambda})' \rightarrow (X^{h, \lambda})' : (f_1, f_2) \rightarrow f_1 - f_2$$

is surjective. This follows from the fact that the transpose of the above mapping is injective and has closed range; we refer to the proof of Proposition 2.2.14 for details. □

13.6.2 Support of hyperfunctions of fast growth

We now define the support of hyperfunctions of fast growth. Recall that, for $K \Subset \mathbb{R}$, we denote by $\mathcal{A}[K] = \mathcal{O}[K]$ the space of germs of

analytic functions on K . Let ω be a weight function satisfying (δ) and $(\epsilon)_0$. For $\Omega \subseteq \overline{\mathbb{R}}$ open we define the (FS) -space

$$\mathcal{O}_{\{\omega\}}(\Omega) := \varprojlim_{\lambda \rightarrow 0^+} \varprojlim_{h \rightarrow 0^+} \varprojlim_{R \rightarrow \infty} \mathcal{O}_\omega^{h,R,\lambda}(\Omega).$$

We say that $f \in \mathcal{A}'_{\{\omega\}}(\mathbb{R})$ *vanishes in* Ω if there is $F \in \mathcal{O}_{\{\omega\}}(\Omega)$ such that $\text{bv}(F) = f$. In view of Theorem 13.5.18, this definition is independent of the chosen analytic representation. The support of f , denoted by $\text{supp } f$, is defined as the complement of the largest open set in which f vanishes. For $K \Subset \overline{\mathbb{R}}$ we set

$$\mathcal{A}'_{\{\omega\}}[K] := \{f \in \mathcal{A}'_{\{\omega\}}(\mathbb{R}) \mid \text{supp } f \subseteq K\}$$

and, for $a, b \in \mathbb{R}$, we employ the short-hand notation

$$\mathcal{A}'_{\{\omega\},a+} = \mathcal{A}'_{\{\omega\}}[[a, \infty]], \quad \mathcal{A}'_{\{\omega\},b-} = \mathcal{A}'_{\{\omega\}}[[-\infty, b]].$$

If $f = \text{bv}(F) \in \mathcal{A}'_{\{\omega\}}[K]$ with $F \in \mathcal{O}_{\{\omega\}}(\overline{\mathbb{R}} \setminus K)$ and J_1, J_2, \dots, J_n is a finite covering of K by open intervals of the extended real line such that their finite end points do not belong to K , Cauchy's integral theorem also gives the contour integral representation (13.6.1) (where $b > 0$ depends on φ). In particular, for $K \Subset \mathbb{R}$, we have that

$$\mathcal{A}'_{\{\omega\}}[K] = \mathcal{A}'[K],$$

as follows from the Silva-Köthe-Grothendieck theorem. In case K is unbounded, we can also represent $\mathcal{A}'_{\{\omega\}}[K]$ as a dual space. We define the (DFS) -space

$$\mathcal{A}_{\{\omega\}}[K] := \varinjlim_{\lambda \rightarrow 0^+} \varinjlim_{h \rightarrow 0^+} \varinjlim_{K \Subset \Omega} \mathcal{A}_\omega^{h,\lambda}(\Omega).$$

Corollary 13.6.3 and the method from Section 13.5.3 yield that:

Theorem 13.6.4. *Let $K \Subset \overline{\mathbb{R}}$ and let ω be a weight function satisfying (δ) and $(\epsilon)_0$. Then, $(\mathcal{A}_{\{\omega\}}[K])' = \mathcal{A}'_{\{\omega\}}[K]$. Furthermore, the sequence*

$$0 \longrightarrow \mathcal{P}_{\{\omega\}} \longrightarrow \mathcal{O}_{\{\omega\}}(\overline{\mathbb{R}} \setminus K) \xrightarrow{\text{bv}} (\mathcal{A}_{\{\omega\}}[K])' \longrightarrow 0$$

is topologically exact.

We also have the ensuing support splitting theorem.

Theorem 13.6.5. *Let $-\infty < a \leq b < \infty$ and let ω be a weight function satisfying (δ) and $(\epsilon)_0$. The sequence*

$$0 \longrightarrow \mathcal{A}'[[a, b]] \longrightarrow \mathcal{A}'_{\{\omega\}, a+} \times \mathcal{A}'_{\{\omega\}, b-} \xrightarrow{T} \mathcal{A}'_{\{\omega\}}(\mathbb{R}) \longrightarrow 0$$

is topologically exact, where $T(f_1, f_2) = f_1 - f_2$.

Proof. Theorem 13.5.18 implies that $\mathcal{A}'[[a, b]] = \mathcal{A}'_{\{\omega\}, a+} \cap \mathcal{A}'_{\{\omega\}, b-}$. The surjectivity of T follows from the fact that its transpose is injective and has dense range (use Theorem 13.6.4). \square

13.7 Spaces of hyperfunctions and ultrahyperfunctions defined via subadditive weight functions

In this section, we briefly indicate how the results from Sections 13.4-13.6 can be extended to include spaces defined in terms of subadditive weight functions. Since the theory and methods are completely analogous to those already developed, we shall omit all proofs. These results will be employed in Chapter 14 to study boundary values of analytic functions in spaces of ultradistributions of exponential type.

13.7.1 Subadditive weight functions

We collect here a number of properties of the weight functions that we shall employ in the rest of this section and Chapter 14. Let ω be a weight function (not necessarily satisfying (13.2.1)). From now on we always assume that $\omega(0) = 0$. We shall use the ensuing conditions [14, 131]:

$$(\alpha)_0 \quad \omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2), \quad t_1, t_2 \geq 0.$$

$$(\gamma) \quad \omega(t) \geq c \log(1 + t) + a, \text{ for some } a \in \mathbb{R} \text{ and } c > 0.$$

In the sequel, we shall refer to condition (13.2.1) as $(\gamma)_0$. We set ${}_\lambda\omega = \lambda\omega$. Let σ be another weight function. The weight functions ω and σ are said to be equivalent, denoted by $\omega \asymp \sigma$, if

$${}_{\lambda_1}\omega(t) - C_1 \leq \sigma(t) \leq {}_{\lambda_2}\omega(t) + C_2, \quad t \geq 0,$$

for some $\lambda_1, \lambda_2, C_1, C_2 > 0$. If ω and σ both satisfy $(\alpha)_0$ and (δ) , then $\omega \asymp \sigma$ if and only if $\omega \sim \sigma$. We point out that subadditivity (condition $(\alpha)_0$) yields the existence of the limit $\lim_{t \rightarrow \infty} \omega(t)/t$ [95, p. 240]. Consequently, we either have $\omega(t) \asymp t$ or $\omega(t) = o(t)$.

The *Young conjugate* of ω is defined as

$$\omega^*(s) := \sup_{t \geq 0} (\omega(t) - ts), \quad s > 0.$$

The function ω^* is convex and decreasing. We set $\omega^*(s) = \omega^*(|s|)$ for $s \in \mathbb{R}$, $s \neq 0$. If $\omega(t) = o(t)$, then $\omega^*(s) < \infty$ for all $s > 0$. Clearly, we have that $({}_\lambda\omega)^* = \lambda\omega^*(\cdot/\lambda)$.

13.7.2 Spaces of analytic functions with rapid decay in strips

Let ω be a weight function. We set $\mathcal{A}_{\lambda\omega}^h = \mathcal{A}_\omega^{h,\lambda}$ and $\|\cdot\|_{\mathcal{A}_{\lambda\omega}^h} = \|\cdot\|_{\mathcal{A}_\omega^{h,\lambda}}$. The basic spaces of entire and analytic functions are now defined as

$$\mathcal{U}_{(\omega)}(\mathbb{C}) := \varprojlim_{h \rightarrow \infty} \mathcal{A}_\omega^{h,h}, \quad \mathcal{A}_{\{\omega\}}(\mathbb{R}) := \varinjlim_{h \rightarrow 0^+} \mathcal{A}_\omega^{h,h}.$$

If $\omega(t) \rightarrow \infty$, Lemma 13.4.1 yields that $\mathcal{U}_{(\omega)}(\mathbb{C})$ is an (FS) -space and that $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ is a (DFS) -space. If ω satisfies both conditions $(\alpha)_0$ and (δ) , then $\mathcal{U}_{(\omega)}(\mathbb{C}) = \mathcal{U}_{(\omega)}(\mathbb{C})$ and $\mathcal{A}_{\{\omega\}}(\mathbb{R}) = \mathcal{A}_{\{\omega\}}(\mathbb{R})$. Let σ be another weight function such that $\omega \asymp \sigma$, then $\mathcal{U}_{(\omega)}(\mathbb{C}) =$

$\mathcal{U}_{(\sigma)}(\mathbb{C})$ and $\mathcal{A}_{\{\omega\}}(\mathbb{R}) = \mathcal{A}_{\{\sigma\}}(\mathbb{R})$. The elements of $\mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$ are called *ultrahyperfunctions of type (ω)* and *hyperfunctions of type $\{\omega\}$* , respectively. Notice that $\mathcal{U}_{(\log(1+t))}(\mathbb{C}) = \mathcal{U}(\mathbb{C})$ is the test function space for the space of tempered ultrahyperfunctions [152, 75, 83].

Remark 13.7.1. Let M_p be a weight sequence satisfying (M.1), (M.2), and (M.4), i.e. the weight sequence $M_p/p!$ satisfies (M.1). Then, there is a weight function ω satisfying $(\alpha)_0, (\gamma)_0$ such that $\omega \asymp M$ [131, Prop. 1.1 and Lemma 5.5]; under these circumstances, we thus have $\mathcal{U}_{(M)}(\mathbb{C}) = \mathcal{U}_{(\omega)}(\mathbb{C})$ and $\mathcal{A}_{\{M\}}(\mathbb{R}) = \mathcal{A}_{\{\omega\}}(\mathbb{R})$. We remark that (M.1) and (M.3) automatically yield (M.4), as shown by Petzsche [130, Prop. 1.1].

Next, we discuss the problem of non-triviality for the spaces $\mathcal{U}_{(\omega)}(\mathbb{C})$ and $\mathcal{A}_{\{\omega\}}(\mathbb{R})$.

Theorem 13.7.2. *Let ω be a weight function satisfying $(\gamma)_0$. Then, $\mathcal{U}_{(\omega)}(\mathbb{C})$ ($\mathcal{A}_{\{\omega\}}(\mathbb{R})$, respectively) is non-trivial if and only if ω satisfies $(\epsilon)_0$ ($(\epsilon)_\infty$, respectively).*

Proof. The conditions are necessary by Proposition 13.4.2. For $\mathcal{A}_{\{\omega\}}(\mathbb{R})$, the sufficiency is a consequence of Proposition 13.3.2. We now consider $\mathcal{U}_{(\omega)}(\mathbb{C})$. By Lemma 13.2.5 we may assume that ω satisfies (δ) . Hence, $\mathcal{U}_{(\omega)}(\mathbb{C}) \subseteq \mathcal{U}_{(\omega)}(\mathbb{C})$ and the result follows from Theorem 13.5.9. \square

Finally, we discuss the Fourier transform. We set $\mathcal{K}_{1,\lambda\omega}^h(\mathbb{R}) = \mathcal{K}_{1,\omega}^{h,\lambda}(\mathbb{R})$ and

$$\mathcal{K}_{1,(\omega)}(\mathbb{R}) := \varprojlim_{h \rightarrow \infty} \mathcal{K}_{1,\omega}^{h,h}(\mathbb{R}), \quad \mathcal{K}_{1,\{\omega\}}(\mathbb{R}) = \varinjlim_{h \rightarrow 0^+} \mathcal{K}_{1,\omega}^{h,h}(\mathbb{R}).$$

In analogy to the terminology from Section 13.4, we call $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ and $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$ the spaces of *ultradistributions of class (ω) of exponential type (of Beurling type)* and *ultradistributions of class $\{\omega\}$ of infra-exponential type (of Roumieu type)*, respectively. If ω satisfies

$(\alpha)_0$, (γ) , and is non-quasianalytic, then we have the dense continuous inclusions $\mathcal{D}_{(\omega)}(\mathbb{R}) \hookrightarrow \mathcal{K}_{1,(\omega)}(\mathbb{R}) \hookrightarrow \mathcal{K}'_{1,(\omega)}(\mathbb{R}) \hookrightarrow \mathcal{D}'_{(\omega)}(\mathbb{R})$; see [14] for the definition of the Beurling-Björck space $\mathcal{D}_{(\omega)}(\mathbb{R})$. In particular, $\mathcal{K}_{1,(\log(1+t))}(\mathbb{R}) = \mathcal{K}_1(\mathbb{R})$ is the space of exponentially rapidly decreasing smooth functions.

Proposition 13.7.3. *Let ω be a weight function satisfying $(\alpha)_0$ and (γ) ($(\gamma)_0$, respectively). Then, the Fourier transform is a topological isomorphism from $\mathcal{U}_{(\omega)}(\mathbb{C})$ onto $\mathcal{K}_{1,(\omega)}(\mathbb{R})$ (from $\mathcal{A}_{\{\omega\}}(\mathbb{R})$ onto $\mathcal{K}_{1,\{\omega\}}(\mathbb{R})$, respectively).*

Proof. This can be shown in the same way as Proposition 13.4.4. □

We define the Fourier transform from $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ onto $\mathcal{U}'_{(\omega)}(\mathbb{C})$ (from $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$ onto $\mathcal{A}'_{\{\omega\}}(\mathbb{R})$, respectively) via duality.

13.7.3 Boundary values

For $0 < b < R$ we set $\mathcal{O}_{\lambda\omega}^{b,R} = \mathcal{O}_{\omega}^{b,R,\lambda}$ and $\mathcal{P}_{\lambda\omega}^R = \mathcal{P}_{\omega}^{R,\lambda}$. Furthermore, we define

$$\mathcal{O}_{(\omega)} := \bigcup_{\lambda, h > 0} \bigcap_{R > h} \mathcal{O}_{\omega}^{h,R,\lambda}, \quad \mathcal{P}_{(\omega)} := \bigcup_{\lambda > 0} \bigcap_{R > 0} \mathcal{P}_{\omega}^{R,\lambda},$$

and

$$\mathcal{O}_{\{\omega\}} := \varprojlim_{\lambda \rightarrow 0^+} \varprojlim_{h \rightarrow 0^+} \varprojlim_{R \rightarrow \infty} \mathcal{O}_{\omega}^{h,R,\lambda}, \quad \mathcal{P}_{\{\omega\}} := \varprojlim_{\lambda \rightarrow 0^+} \varprojlim_{R \rightarrow \infty} \mathcal{P}_{\omega}^{R,\lambda}.$$

If $\omega(t) \rightarrow \infty$, Lemma 13.5.1 implies that $\mathcal{O}_{\{\omega\}}$ and $\mathcal{P}_{\{\omega\}}$ are (FS) -spaces. The boundary value mappings $\text{bv} : \mathcal{O}_{(\omega)} \rightarrow \mathcal{U}'_{(\omega)}(\mathbb{C})$ and $\text{bv} : \mathcal{O}_{\{\omega\}} \rightarrow \mathcal{A}'_{\{\omega\}}(\mathbb{R})$ are well-defined. Taking Remark 13.3.3 into account, Propositions 13.5.2 and 13.5.3 yield the following corollaries.

Corollary 13.7.4. *Let $0 < k < b < h < R$ and let ω be a weight function satisfying $(\alpha)_0$.*

- (i) If ω satisfies (γ) , then for every $f \in (\mathcal{A}_{\omega}^{k,\lambda})'$ there is $F \in \mathcal{O}_{\omega}^{b,R,4\lambda+2c^{-1}}$ such that $\text{bv}(F) = f$ on $\mathcal{A}_{\omega}^{h,4\lambda+4c^{-1}}$.
- (ii) If ω satisfies $(\gamma)_0$, then for every $f \in (\mathcal{A}_{\omega}^{k,\lambda})'$ there is $F \in \mathcal{O}_{\omega}^{b,R,4\lambda}$ such that $\text{bv}(F) = f$ on $\mathcal{A}_{\omega}^{h,8\lambda}$.

Corollary 13.7.5. *Let $0 < b < h < R$ and let ω be a weight function satisfying $(\alpha)_0$.*

- (i) If ω satisfies (γ) and $F \in \mathcal{O}_{\omega}^{b,R,\lambda}$ is such that $\text{bv}(F) = 0$ on $\mathcal{A}_{\omega}^{h,\lambda+2c^{-1}}$, then $F \in \mathcal{P}_{\omega}^{R,4\lambda+8c^{-1}}$.
- (ii) If ω satisfies $(\gamma)_0$ and $F \in \mathcal{O}_{\omega}^{b,R,\lambda}$ is such that $\text{bv}(F) = 0$ on $\mathcal{A}_{\omega}^{h,2\lambda}$, then $F \in \mathcal{P}_{\omega}^{R,8\lambda}$.

By using exactly the same technique as in Sections 13.5.2 and 13.5.3 and by applying Corollaries 13.7.4 and 13.7.5 instead of Corollaries 13.5.4 and 13.5.5) one can show the following theorem.

Theorem 13.7.6. *Let ω be a weight function satisfying $(\alpha)_0$.*

- (i) If ω satisfies (γ) , then the sequence

$$0 \longrightarrow \mathcal{P}_{(\omega)} \longrightarrow \mathcal{O}_{(\omega)} \xrightarrow{\text{bv}} \mathcal{U}'_{(\omega)}(\mathbb{C}) \longrightarrow 0$$

is exact.

- (ii) If ω satisfies $(\gamma)_0$, then the sequence

$$0 \longrightarrow \mathcal{P}_{\{\omega\}} \longrightarrow \mathcal{O}_{\{\omega\}} \xrightarrow{\text{bv}} \mathcal{A}'_{\{\omega\}}(\mathbb{R}) \longrightarrow 0$$

is topologically exact.

Let us briefly discuss the notion of support. For $\Omega \subseteq \overline{\mathbb{R}}$ open we set $\mathcal{O}_{\lambda\omega}^{h,R}(\Omega) = \mathcal{O}_{\omega}^{h,R,\lambda}(\Omega)$ and define

$$\mathcal{O}_{(\omega)}(\Omega + i\mathbb{R}) := \bigcup_{\lambda, h > 0} \bigcap_{R > h} \mathcal{O}_{\omega}^{h,R,\lambda}(\Omega)$$

and

$$\mathcal{O}_{\{\omega\}}(\Omega) := \varprojlim_{\lambda \rightarrow 0^+} \varprojlim_{h \rightarrow 0^+} \varprojlim_{R \rightarrow \infty} \mathcal{O}_{\omega}^{h,R,\lambda}(\Omega).$$

We suppose that ω satisfies $(\alpha)_0$ and (γ) ($(\gamma)_0$, respectively). Vanishing of $f \in \mathcal{U}'_{(\omega)}(\mathbb{C})$ ($f \in \mathcal{A}'_{\{\omega\}}(\mathbb{R})$, respectively) in an open set $\Omega \subseteq \overline{\mathbb{R}}$ means that there is $F \in \mathcal{O}_{(\omega)}(\Omega + i\mathbb{R})$ ($F \in \mathcal{O}_{\{\omega\}}(\Omega)$, respectively) such that $\text{bv}(F) = f$. The definition of $\text{supp}_{\overline{\mathbb{R}}} f$ ($\text{supp } f$, respectively) should now be clear. We shall adopt the same kind of notations as in Section 13.6 for the rest of the spaces defined there by simply replacing ω by ω . Furthermore, all results from that section remain valid in our new context; in fact, due to Remark 13.3.3 and the general formulation of Proposition 13.6.2, the same proofs apply here. In particular, we state the support splitting theorem for future reference.

Theorem 13.7.7. *Let $-\infty < a \leq b < \infty$ and let ω be a weight function satisfying $(\alpha)_0$.*

(i) *If ω satisfies (γ) , then the sequence*

$$0 \longrightarrow \mathcal{O}'[[a, b] + i\mathbb{R}] \longrightarrow \mathcal{U}'_{(\omega),a+} \times \mathcal{U}'_{(\omega),b-} \xrightarrow{T} \mathcal{U}'_{(\omega)}(\mathbb{C}) \longrightarrow 0$$

is exact, where $T(f_1, f_2) = f_1 - f_2$.

(ii) *If ω satisfies $(\gamma)_0$, then the sequence*

$$0 \longrightarrow \mathcal{A}'[[a, b]] \longrightarrow \mathcal{A}'_{\{\omega\},a+} \times \mathcal{A}'_{\{\omega\},b-} \xrightarrow{T} \mathcal{A}'_{\{\omega\}}(\mathbb{R}) \longrightarrow 0$$

is topologically exact, where $T(f_1, f_2) = f_1 - f_2$.

Chapter 14

Boundary values of analytic functions in ultradistribution spaces of exponential type

14.1 Introduction

The study of boundary values of analytic functions in spaces of (ultra)distributions has a long and rich tradition that goes back to the pioneer works of Köthe [98, 99]; we refer to the books [25] and [24] for a detailed account on this subject in the distribution and ultradistribution case, respectively. The goal of this chapter is to investigate boundary values of analytic functions in the ultradistribution spaces of (infra-)exponential type $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ and $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$, where ω is a subadditive weight function which may be of quasianalytic type (cf. Section 13.7).

Let us briefly comment on the methods to be employed in this chapter (we discuss here the case $\omega(t) = o(t)$; the case $\omega(t) \asymp t$ goes along the same lines but is in fact simpler). We start with

a discussion about the Laplace transform of (ultra)hyperfunctions of fast growth supported in a left-bounded interval. We show that for any $f \in \mathcal{U}'_{(\omega),a+}$ ($f \in \mathcal{A}'_{\{\omega\},a+}$, respectively), $a \geq 0$ fixed, its Laplace transform $\mathcal{L}\{f, \cdot\}$ is an analytic function in the upper-half plane satisfying certain growth conditions with respect to ω such that

$$\lim_{\eta \rightarrow 0^+} \mathcal{L}\{f; \cdot + i\eta\} = \mathcal{F}^{-1}(f)$$

in $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ (in $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$, respectively). Of course, such results remain valid for (ultra)hyperfunctions of fast growth supported in a right-bounded interval if one replaces the upper-half plane by the lower one. In order to show that any ultradistribution of (infra-)exponential type can be represented as the boundary value of an analytic function, we then use the following simple but beautiful idea due to Carleman [23]: Let $g \in \mathcal{K}'_{1,(\omega)}(\mathbb{R})$ ($g \in \mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$, respectively) be arbitrary and set $f = \widehat{g}$. By Theorem 13.7.7 we can decompose f as $f = f_+ - f_-$, where f_+ and f_- belong to $\mathcal{U}'_{(\omega)}(\mathbb{C})$ ($\mathcal{A}'_{\{\omega\}}(\mathbb{R})$, respectively) and are supported on the positive and negative half-axis, respectively. Now define

$$G(\zeta) = \begin{cases} \mathcal{L}\{f_+; \zeta\}, & \text{Im } \zeta > 0, \\ \mathcal{L}\{f_-; \zeta\}, & \text{Im } \zeta < 0. \end{cases}$$

Then,

$$g = \lim_{\eta \rightarrow 0^+} G(\cdot + i\eta) - G(\cdot - i\eta)$$

in $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ (in $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$, respectively). In order to prove that any analytic function satisfying adequate growth conditions admits a boundary value as an ultradistribution of (infra-)exponential type, we characterize the test function spaces $\mathcal{K}_{1,(\omega)}(\mathbb{R})$ and $\mathcal{K}_{1,\{\omega\}}(\mathbb{R})$ in terms of almost analytic extensions [131, 24, 78] and then use the Stokes theorem; this method seems to go back to Hörmander [79, second proof of Thm. 3.1.11]. The construction of such almost analytic extensions shall be achieved by an explicit formula that

is a slight modification of an ingenious formula due to Petzsche and Vogt [131]. By combining these results with Theorems 13.5.14 and 13.5.18 (for $\omega(t) = t$), we will be able to represent the spaces $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ and $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$ as the quotient of certain weighted spaces of analytic functions.

This chapter is organized as follows. In Section 14.2, we define the Laplace transform of (ultra)hyperfunctions of fast growth while Section 14.3 is devoted to the study of boundary values of analytic functions in the spaces $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ and $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$.

14.2 Laplace transform

In this auxiliary section, we study the Laplace transform of (ultra)hyperfunctions of fast growth with support in a proper closed interval I of $\overline{\mathbb{R}}$. Let us first fix some notation. We write $z = x + iy \in \mathbb{C}$ and $\zeta = \xi + i\eta \in \mathbb{C}$ for complex variables and set

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

The following condition for weight functions plays a role below:

$$(NA) \quad \omega(t) = o(t).$$

From now on we assume that ω is a weight function satisfying $(\alpha)_0$ and (γ) . Let $f \in \mathcal{U}'_{(\omega)}[I + i\mathbb{R}]$. We define the Laplace transform of f as

$$\mathcal{L}\{f; \zeta\} := -\frac{1}{2\pi} \int_{\Gamma^b(J)} F(z) e^{iz\zeta} dz, \quad (14.2.1)$$

for $\zeta \in \mathbb{C}$ in a suitable domain to be specified below and where $F \in \mathcal{O}^{h,R}_{(\omega)}(\mathbb{R} \setminus I)$ is an analytic representation of f ($\text{bv}(F) = f$), $0 < h < b < R$, and J is an open interval in $\overline{\mathbb{R}}$ such that $I \Subset J$. This definition is independent of the chosen representative of F . In the rest of the discussion, we distinguish three cases.

Case I: *a bounded interval* $I = [-a, a]$, $0 \leq a < \infty$. In this case, (14.2.1) is defined for all $\zeta \in \mathbb{C}$. In fact, $\mathcal{L}\{f; \cdot\}$ is an entire function that satisfies the estimate: There is $h > 0$ such that for every $\varepsilon > 0$

$$\sup_{\zeta \in \mathbb{C}} |\mathcal{L}\{f; \zeta\}| e^{-(a+\varepsilon)|\eta| - (h+\varepsilon)|\xi|} < \infty. \quad (14.2.2)$$

Conversely, if G is an entire function that satisfies (14.2.2) for some $h > 0$ and every $\varepsilon > 0$, then there is $f \in \mathcal{O}'[I + i[-h, h]] \subset \mathcal{U}'_{(\omega)}[I + i\mathbb{R}]$ such that $G = \mathcal{L}\{f; \cdot\}$, as follows from the Paley-Wiener-Schwartz theorem for analytic functionals [117, Thm. 2.5.2]. If ω satisfies $(\gamma)_0$ and $f \in \mathcal{A}'_{\{\omega\}}[I] = \mathcal{A}'[I]$, then $\mathcal{L}\{f; \cdot\}$ satisfies (14.2.2) for every $h, \varepsilon > 0$, and the converse holds true: If an entire function G satisfies (14.2.2) for every $h, \varepsilon > 0$, then there is $f \in \mathcal{A}'[I]$ such that $G = \mathcal{L}\{f; \cdot\}$.

Case II: *a left-bounded interval* $I = [-a, \infty]$, $0 \leq a < \infty$. As was pointed out in Subsection 13.7.1, either ω satisfies (NA) or $\omega(t) \asymp t$. First assume that ω satisfies (NA) . Then, $\mathcal{L}\{f; \cdot\}$ is analytic in the upper half-plane $\text{Im } \zeta = \eta > 0$ and satisfies the bound: There are $\lambda, h > 0$ such that for every $\varepsilon > 0$

$$\sup_{\eta > 0} |\mathcal{L}\{f; \zeta\}| e^{-(a+\varepsilon)\eta - (h+\varepsilon)|\xi| - \lambda\omega^*(\eta/\lambda)} < \infty. \quad (14.2.3)$$

Moreover, the Laplace transform has as boundary value the inverse Fourier transform of f , namely,

$$\lim_{\eta \rightarrow 0^+} \mathcal{L}\{f; \cdot + i\eta\} = \mathcal{F}^{-1}(f) \quad \text{in } \mathcal{K}'_{1,(\omega)}(\mathbb{R}).$$

If ω satisfies $(\gamma)_0$ and $f \in \mathcal{A}'_{\{\omega\}}[I]$, then $\mathcal{L}\{f; \cdot\}$ satisfies (14.2.3) for every $\lambda, h, \varepsilon > 0$ and

$$\lim_{\eta \rightarrow 0^+} \mathcal{L}\{f; \cdot + i\eta\} = \mathcal{F}^{-1}(f) \quad \text{in } \mathcal{K}'_{1,\{\omega\}}(\mathbb{R}).$$

Next, assume $\omega(t) \asymp t$ and, therefore, $\mathcal{K}_{1,(\omega)}(\mathbb{R}) = \mathcal{U}_{(t)}(\mathbb{C})$ and $\mathcal{K}_{1,\{\omega\}}(\mathbb{R}) = \mathcal{A}_{\{t\}}(\mathbb{R})$ are invariant under the Fourier transform. If

$f \in \mathcal{U}'_{\{t\}}[I + i\mathbb{R}]$, then there are $\lambda, h > 0$ such that $\mathcal{L}\{f; \cdot\}$ is a holomorphic function in the half-plane $\text{Im } \zeta = \eta > \lambda$ and satisfies

$$\sup_{\eta > \lambda} |\mathcal{L}\{f; \zeta\}| e^{-(a+\varepsilon)\eta - (h+\varepsilon)|\xi|} < \infty. \quad (14.2.4)$$

Set

$$G(\zeta) = \begin{cases} \mathcal{L}\{f; \zeta\}, & \eta > \lambda, \\ 0, & \eta < -\lambda. \end{cases}$$

Then, $\text{bv}(G) = \mathcal{F}^{-1}(f)$ in $\mathcal{U}'_{\{t\}}(\mathbb{C})$ in the sense of Section 13.7.3. If $f \in \mathcal{A}'_{\{t\}}[I]$, then $\mathcal{L}\{f; \cdot\}$ is analytic in the upper half-plane and satisfies (14.2.4) for every $\lambda, h, \varepsilon > 0$. Furthermore, $\text{bv}(G) = \mathcal{F}^{-1}(f)$ in $\mathcal{A}'_{\{t\}}(\mathbb{R})$, where now

$$G(\zeta) = \begin{cases} \mathcal{L}\{f; \zeta\}, & \eta > 0, \\ 0, & \eta < 0. \end{cases}$$

Case III: a *right-bounded interval* $I = [-\infty, a]$, $0 \leq a < \infty$. Here the treatment is completely analogous to case II but with the upper half-planes replaced by lower ones.

14.3 Boundary values

We now study the boundary values of analytic functions in the spaces $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ and $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$. Our first aim is to obtain almost analytic extensions of elements of $\mathcal{K}_{1,(\omega)}(\mathbb{R})$ and $\mathcal{K}_{1,\{\omega\}}(\mathbb{R})$. We start with the following result from [131] that allows one to replace a weight function by an equivalent one enjoying much better regularity properties.

Lemma 14.3.1. ([131, Prop. 1.2]) *Let ω be a weight function satisfying $(\alpha)_0$, (γ) , and (NA) . Then, there is a weight function σ with $\omega \asymp \sigma$ satisfying the following conditions:*

(i) $\sigma \in C^\infty((0, \infty))$ and $\lim_{t \rightarrow 0^+} \sigma'(t) = \infty$.

(ii) σ is strictly concave.

(iii) $\lim_{t \rightarrow \infty} \sigma'(t) = 0$.

(iv) $\liminf_{t \rightarrow \infty} |\sigma''(t)|t^2 > 0$.

If ω satisfies $(\gamma)_0$, then (iv) may be replaced by the stronger property

(iv)' $\lim_{t \rightarrow \infty} |\sigma''(t)|t^2 = \infty$.

The next key lemma is a modification of [131, Prop. 2.2].

Lemma 14.3.2. *Let $0 < k < h$. Let ω be a weight function satisfying the conditions (i)-(iv) of Lemma 14.3.1 and let σ be another weight function such that $\sigma(t) \leq \omega(t)$ for all $t \geq 0$ and*

$$\int_0^\infty t e^{\omega(t)-\sigma(t)} dt < \infty. \quad (14.3.1)$$

Then, for every $\varphi \in \mathcal{A}_\sigma^h$ there is $\Psi \in C^\infty(\mathbb{C})$ with $\Psi|_{\mathbb{R}} = \hat{\varphi}$ such that

$$|\bar{\partial}\Psi(\zeta)| \leq \|\varphi\|_{\mathcal{A}_\omega^h} e^{-k|\xi|} |(\omega^*)''(\eta)| e^{-\omega^*(\eta)}, \quad \zeta \in \mathbb{C} \setminus \mathbb{R}, \quad (14.3.2)$$

and

$$|\Psi(\zeta)| \leq C \|\varphi\|_{\mathcal{A}_\sigma^h} e^{-k|\xi|}, \quad \zeta \in \mathbb{C}, \quad (14.3.3)$$

where

$$C = 2 \int_0^\infty e^{\omega(t)-\sigma(t)} dt.$$

Proof. The assumptions on ω imply that ω' is a smooth bijection on $(0, \infty)$. Set $H = (\omega')^{-1} \in C^\infty((0, \infty))$ and observe that $\omega^*(s) = \omega(H(s)) - sH(s)$ for all $s > 0$. By differentiating the latter equality we obtain that $(\omega^*)' = -H$. We set $H(s) = H(|s|)$ for $s \in \mathbb{R}$, $s \neq 0$. Furthermore, since ω is concave and increasing, we have that

$$t\omega'(t) \leq \omega(t), \quad t \geq 0. \quad (14.3.4)$$

The proof is based on the following representation of the Fourier transform of φ (cf. the proof of Proposition 13.4.4)

$$\widehat{\varphi}(\xi) = \begin{cases} \int_{-\infty}^{\infty} \varphi(x + ik)e^{-i(x+ik)\xi} dx, & \xi \leq 0, \\ \int_{-\infty}^{\infty} \varphi(x - ik)e^{-i(x-ik)\xi} dx, & \xi \geq 0. \end{cases}$$

For $\zeta \in \mathbb{C} \setminus \mathbb{R}$ we define

$$\Psi(\zeta) = \begin{cases} \int_{-H(\eta)}^{H(\eta)} \varphi(x + ik)e^{-i(x+ik)\zeta} dx, & \xi \leq 0, \\ \int_{-H(\eta)}^{H(\eta)} \varphi(x - ik)e^{-i(x-ik)\zeta} dx, & \xi \geq 0, \end{cases}$$

and $\Psi(\xi) = \widehat{\varphi}(\xi)$ for $\xi \in \mathbb{R}$. Clearly, $\Psi \in C^\infty(\mathbb{C} \setminus \mathbb{R})$. By employing (14.3.1) and (14.3.4) one can prove that $\Psi \in C^1(\mathbb{C})$ and $\bar{\partial}\Psi(\xi) = 0$ for all $\xi \in \mathbb{R}$, whence $\Psi \in C^\infty(\mathbb{C})$. We now show (14.3.2). Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$ and assume that $\xi \leq 0$; the case $\xi \geq 0$ is similar. The remarks given at the beginning of the proof yield that

$$\begin{aligned} |\bar{\partial}\Psi(\zeta)| &\leq \frac{1}{2} |H'(\eta)(\varphi(H(\eta) + ik)e^{-i(H(\eta)+ik)\zeta} \\ &\quad + \varphi(-H(\eta) + ik)e^{-i(-H(\eta)+ik)\zeta})| \\ &\leq \|\varphi\|_{\mathcal{A}_\omega^h} e^{-k|\xi|} |H'(\eta)| e^{-\omega(H(\eta))+H(\eta)|\eta|} \\ &= \|\varphi\|_{\mathcal{A}_\omega^h} e^{-k|\xi|} |(\omega^*)''(\eta)| e^{-\omega^*(\eta)}. \end{aligned}$$

It remains to establish (14.3.3). Since Ψ is continuous, it suffices to show this for $\zeta \in \mathbb{C} \setminus \mathbb{R}$. We assume that $\xi \leq 0$; the case $\xi \geq 0$ is similar. Then,

$$|\Psi(\zeta)| \leq 2\|\varphi\|_{\mathcal{A}_\sigma^h} e^{-k|\xi|} \int_0^{H(\eta)} e^{-\sigma(x)+x|\eta|} dx.$$

For $0 < x < H(\eta)$ we have that $\omega'(x) > |\eta|$. By applying (14.3.4) we obtain that

$$\int_0^{H(\eta)} e^{-\sigma(x)+x|\eta|} dx \leq \int_0^{H(\eta)} e^{-\sigma(x)+x\omega'(x)} dx \leq \int_0^\infty e^{\omega(x)-\sigma(x)} dx. \quad \square$$

Corollary 14.3.3. *Let ω be a weight function satisfying $(\alpha)_0$ and (NA) .*

- (i) *If ω satisfies (γ) , then for every $\psi \in \mathcal{K}_{1,(\omega)}(\mathbb{R})$ and every $\lambda, h > 0$ there is $\Psi \in C^\infty(\mathbb{C})$ with $\Psi|_{\mathbb{R}} = \psi$ satisfying the bounds*

$$\sup_{\zeta \in \mathbb{C} \setminus \mathbb{R}} |\bar{\partial}\Psi(\zeta)| e^{h|\xi| + \lambda\omega^*(\eta/\lambda)} < \infty, \quad \sup_{\zeta \in \mathbb{C}} |\Psi(\zeta)| e^{h|\xi|} < \infty. \quad (14.3.5)$$

- (ii) *If ω satisfies $(\gamma)_0$, then for every $\psi \in \mathcal{K}_{1,\{\omega\}}(\mathbb{R})$ there are $\lambda, h > 0$ and $\Psi \in C^\infty(\mathbb{C})$ with $\Psi|_{\mathbb{R}} = \psi$ satisfying the inequalities (14.3.5).*

Proof. We may assume that ω satisfies the conditions (i)-(iv) from Lemma 14.3.1 (conditions (i)-(iv)' for (ii)). In [131, Lemma 2.3] it is shown that there is $\varepsilon > 0$ such that $\sup_{s>0} |(\omega^*)''(s)| e^{-\varepsilon\omega^*(s)} < \infty$ and that if, in addition, ω satisfies (iv)', then the latter inequality holds for every $\varepsilon > 0$. Therefore, the result follows by applying Lemma 14.3.2 to the weight $\mu\omega$, for a suitable $\mu > 0$, and $\varphi = \mathcal{F}^{-1}(\psi)$. \square

The next result gives a sufficient condition for the existence of boundary values of analytic functions in the ultradistribution spaces of (infra-)exponential type $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ and $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$.

Proposition 14.3.4. *Let ω be a weight function satisfying $(\alpha)_0$ and (NA) .*

- (i) *If ω satisfies (γ) , then every $G \in \mathcal{O}(T_+^R)$ satisfying*

$$\sup_{\zeta \in T_+^R} |G(\zeta)| e^{-h|\xi| - \lambda\omega^*(\eta/\lambda)} < \infty \quad (14.3.6)$$

for some $\lambda, h > 0$, has a boundary value in $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$, that is, there is $g \in \mathcal{K}'_{1,(\omega)}(\mathbb{R})$, such that

$$g = \lim_{\eta \rightarrow 0^+} G(\cdot + i\eta) \quad \text{in } \mathcal{K}'_{1,(\omega)}(\mathbb{R}).$$

(ii) If ω satisfies $(\gamma)_0$, then every $G \in \mathcal{O}(T_+^R)$ satisfying the inequality (14.3.6) for every $\lambda, h > 0$, has a boundary value in $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$.

Proof. We only treat (ii); the case (i) is similar. Since the space $\mathcal{K}_{1,\{\omega\}}(\mathbb{R})$ is Montel, it suffices to show that

$$\lim_{\eta \rightarrow 0^+} \int_{-\infty}^{\infty} G(\xi + i\eta) \psi(\xi) d\xi$$

exists and is finite for every $\psi \in \mathcal{K}_{1,\{\omega\}}(\mathbb{R})$. By Corollary 14.3.3 there is $\Psi \in C^\infty(\mathbb{C})$ with $\Psi|_{\mathbb{R}} = \psi$ satisfying the inequalities (14.3.5) for some $\lambda, h > 0$. Choose $0 < L < R$ and fix $0 < \eta < R - L$. Define $\tilde{G}(\xi + iv) = G(\xi + i(\eta + v))\Psi(\xi + iv)$ for $\xi + iv \in \mathbb{R} + i[-L, L]$. By applying the Stokes theorem to the rectangle $(-N, N) + i(0, L)$, $N > 0$ arbitrary but fixed, and \tilde{G} we obtain that

$$\begin{aligned} \int_{-N}^N G(\xi + i\eta) \psi(\xi) d\xi &= \int_{-N}^N G(\xi + i(\eta + L)) \Psi(\xi + iL) d\xi \\ &\quad - \int_0^L G(N + i(\eta + v)) \Psi(N + iv) dv \\ &\quad + \int_0^L G(-N + i(\eta + v)) \Psi(-N + iv) dv \\ &\quad + 2i \int_{-N}^N \int_0^L G(\xi + i(\eta + v)) \bar{\partial} \Psi(\xi + iv) dv d\xi. \end{aligned}$$

The second and third integral in the right-hand side tend to zero as $N \rightarrow \infty$. Hence,

$$\begin{aligned} &\int_{-\infty}^{\infty} G(\xi + i\eta) \psi(\xi) d\xi \\ &= \int_{-\infty}^{\infty} G(\xi + i(\eta + L)) \Psi(\xi + iL) d\xi \\ &\quad + 2i \int_{-\infty}^{\infty} \int_0^L G(\xi + i(\eta + v)) \bar{\partial} \Psi(\xi + iv) dv d\xi. \end{aligned}$$

Finally, Lebesgue's dominated convergence theorem implies that

$$\begin{aligned} & \lim_{\eta \rightarrow 0^+} \int_{-\infty}^{\infty} G(\xi + i\eta) \psi(\xi) d\xi \\ &= \int_{-\infty}^{\infty} G(\xi + iL) \Psi(\xi + iL) d\xi \\ &+ 2i \int_{-\infty}^{\infty} \int_0^L G(\xi + iv) \bar{\partial} \Psi(\xi + iv) dv d\xi < \infty. \end{aligned}$$

□

We now have all the necessary tools to express $\mathcal{K}'_{1,(\omega)}(\mathbb{R})$ and $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$ as quotients of spaces of analytic functions. Let $a \geq 0$ and let ω be a weight function satisfying $(\alpha)_0$, (γ) , and (NA) . We introduce the space $\mathcal{O}_{\omega,\lambda}^{\exp,h,a}(\mathbb{C} \setminus \mathbb{R})$ consisting of all $G \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ satisfying

$$\sup_{\zeta \in \mathbb{C} \setminus \mathbb{R}} |G(\zeta)| e^{-(a+\varepsilon)|\eta| - (h+\varepsilon)|\xi| - \lambda \omega^*(\eta/\lambda)} < \infty$$

for every $\varepsilon > 0$. Set

$$\mathcal{O}_{(\omega)}^{(\exp),a}(\mathbb{C} \setminus \mathbb{R}) := \bigcup_{\lambda, h > 0} \mathcal{O}_{\omega,\lambda}^{\exp,h,a}(\mathbb{C} \setminus \mathbb{R})$$

and

$$\mathcal{O}_{\{\omega\}}^{\{\exp\},a}(\mathbb{C} \setminus \mathbb{R}) := \bigcap_{\lambda, h > 0} \mathcal{O}_{\omega,\lambda}^{\exp,h,a}(\mathbb{C} \setminus \mathbb{R}).$$

We define the boundary value mapping as follows

$$\text{bv} : \mathcal{O}_{(\omega)}^{(\exp),a}(\mathbb{C} \setminus \mathbb{R}) \rightarrow \mathcal{K}'_{1,(\omega)}(\mathbb{R}) : G \rightarrow \lim_{\eta \rightarrow 0^+} G(\cdot + i\eta) - G(\cdot - i\eta)$$

Proposition 14.3.4 guarantees that bv is well-defined. Moreover, if ω satisfies $(\gamma)_0$, then $\text{bv}(\mathcal{O}_{\{\omega\}}^{\{\exp\},a}(\mathbb{C} \setminus \mathbb{R})) \subseteq \mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$. Next, we write $\mathcal{O}^{\exp,h,a}(\mathbb{C})$ for the space consisting of all entire functions $G \in \mathcal{O}(\mathbb{C})$ such that

$$\sup_{\zeta \in \mathbb{C}} |G(\zeta)| e^{-(a+\varepsilon)|\eta| - (h+\varepsilon)|\xi|} < \infty$$

for every $\varepsilon > 0$ and set

$$\mathcal{O}^{(\exp),a}(\mathbb{C}) := \bigcup_{h > 0} \mathcal{O}^{\exp,h,a}(\mathbb{C}), \quad \mathcal{O}^{\{\exp\},a}(\mathbb{C}) := \bigcap_{h > 0} \mathcal{O}^{\exp,h,a}(\mathbb{C}).$$

Theorem 14.3.5. *Let $a \geq 0$ and let ω be a weight function satisfying $(\alpha)_0$ and (NA) .*

(i) *If ω satisfies (γ) , then the sequence*

$$0 \longrightarrow \mathcal{O}^{(\exp),a}(\mathbb{C}) \longrightarrow \mathcal{O}_{\{\omega\}}^{(\exp),a}(\mathbb{C} \setminus \mathbb{R}) \xrightarrow{\text{bv}} \mathcal{K}'_{1,\{\omega\}}(\mathbb{R}) \longrightarrow 0$$

is exact.

(ii) *If ω satisfies $(\gamma)_0$, then the sequence*

$$0 \longrightarrow \mathcal{O}^{\{\exp\},a}(\mathbb{C}) \longrightarrow \mathcal{O}_{\{\omega\}}^{\{\exp\},a}(\mathbb{C} \setminus \mathbb{R}) \xrightarrow{\text{bv}} \mathcal{K}'_{1,\{\omega\}}(\mathbb{R}) \longrightarrow 0$$

is exact.

Proof. We only treat (ii); the case (i) is similar. The equality $\ker \text{bv} = \mathcal{O}^{\{\exp\},a}(\mathbb{C})$ follows from Theorem 13.5.18 with $\omega(t) = t$. We now show that the boundary value mapping is surjective. Let $g \in \mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$ and set $f = \widehat{g} \in \mathcal{A}'_{\{\omega\}}(\mathbb{R})$. By Theorem 13.7.7 there are $f_+ \in \mathcal{A}'_{\{\omega\},(-a)_+}$ and $f_- \in \mathcal{A}'_{\{\omega\},a_-}$ such that $f = f_+ - f_-$. Define

$$G(\zeta) = \begin{cases} \mathcal{L}\{f_+; \zeta\}, & \eta > 0, \\ \mathcal{L}\{f_-; \zeta\}, & \eta < 0. \end{cases}$$

From the discussion in Section 14.2 on the Laplace transform, it is clear that $G \in \mathcal{O}_{\{\omega\}}^{\{\exp\},a}(\mathbb{C} \setminus \mathbb{R})$ and $\text{bv}(G) = g$. \square

As an application of Theorem 14.3.5, we now characterize in a precise fashion those analytic functions in the upper half-plane that are the Laplace transform of an ultrahyperfunction of class (ω) and a hyperfunction of class $\{\omega\}$, respectively, supported in a fixed half-axis.

Theorem 14.3.6. *Let $a \geq 0$. Let ω be a weight function satisfying $(\alpha)_0$ and (NA) , and suppose that G is an analytic function in the upper half-plane.*

(i) If ω satisfies (γ) , then G satisfies the estimate

$$\sup_{\eta > 0} |G(\zeta)| e^{-(a+\varepsilon)\eta - (h+\varepsilon)|\xi| - \lambda\omega^*(\eta/\lambda)} < \infty \quad (14.3.7)$$

for some $h, \lambda > 0$ and every $\varepsilon > 0$ if and only if there is $f \in \mathcal{U}'_{(\omega),(-a)+}$ such that $G = \mathcal{L}\{f; \cdot\}$.

(ii) If ω satisfies $(\gamma)_0$, then G satisfies (14.3.7) for every $h, \lambda, \varepsilon > 0$ if and only if there is $f \in \mathcal{A}'_{\{\omega\},(-a)+}$ such that $G = \mathcal{L}\{f; \cdot\}$.

Proof. We only treat (ii); the case (i) is similar. It has already been pointed out in Section 14.2 that the Laplace transform of an element of $\mathcal{A}'_{\{\omega\},(-a)+}$ satisfies the required bounds. Conversely, let G be an analytic function in the upper half-plane satisfying (14.3.7) for every $h, \lambda, \varepsilon > 0$. By Proposition 14.3.4 there is $g \in \mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$ such that $g = \lim_{\eta \rightarrow 0+} G(\cdot + i\eta)$ in $\mathcal{K}'_{1,\{\omega\}}(\mathbb{R})$. Let $f = \widehat{g} \in \mathcal{A}'_{\{\omega\}}(\mathbb{R})$. By Theorem 13.7.7 there are $f_+ \in \mathcal{A}'_{\{\omega\},(-a)+}$ and $f_- \in \mathcal{A}'_{\{\omega\},a-}$ such that $f = f_+ - f_-$. Define

$$\widetilde{G}(\zeta) = \begin{cases} \mathcal{L}\{f_+; \zeta\}, & \eta > 0, \\ \mathcal{L}\{f_-; \zeta\}, & \eta < 0. \end{cases}$$

Notice that $\widetilde{G} \in \mathcal{O}^{\{\exp\},a}_{\{\omega\}}(\mathbb{C} \setminus \mathbb{R})$ and $\text{bv}(\widetilde{G}) = g$. Hence, by Theorem 14.3.5, there is $G_c \in \mathcal{O}^{\{\exp\},a}(\mathbb{C})$ such that $G = \widetilde{G} + G_c$ in the upper half-plane. Since there is $f_c \in \mathcal{A}'[[-a, a]]$ such that $G_c = \mathcal{L}\{f_c; \cdot\}$ (Case I in Section 14.2), we conclude that $G = \mathcal{L}\{f_+ + f_c; \cdot\}$. \square

If ω satisfies $(\alpha)_0$ but not (NA) , we must have $\omega(t) \asymp t$ and, thus, $\mathcal{K}_{1,(\omega)}(\mathbb{R}) = \mathcal{U}_{(t)}(\mathbb{C})$ and $\mathcal{K}_{1,\{\omega\}}(\mathbb{R}) = \mathcal{A}_{\{t\}}(\mathbb{R})$. In this case, the counterparts of Theorems 14.3.5 and 14.3.6 go back to the work of Silva and Morimoto [116, 152]. For the sake of completeness, we end this chapter by stating these theorems. For $a \geq 0$ we define $\mathcal{O}^{\exp,h,a}(\mathbb{C} \setminus \overline{T^\lambda})$ as the space consisting of all $G \in \mathcal{O}(\mathbb{C} \setminus \overline{T^\lambda})$ such that

$$\sup_{\zeta \in \mathbb{C} \setminus \overline{T^\lambda}} |G(\zeta)| e^{-(a+\varepsilon)|\eta| - (h+\varepsilon)|\xi|} < \infty$$

for every $\varepsilon > 0$. Set

$$\mathcal{O}^{(\text{exp}),a} := \bigcup_{\lambda, h > 0} \mathcal{O}^{\text{exp},h,a}(\mathbb{C} \setminus \overline{T^\lambda})$$

and

$$\mathcal{O}^{\{\text{exp}\},a}(\mathbb{C} \setminus \mathbb{R}) := \bigcap_{\lambda, h > 0} \mathcal{O}^{\text{exp},h,a}(\mathbb{C} \setminus \overline{T^\lambda}).$$

The proofs of the ensuing two results go along the same lines as those of Theorems 14.3.5 and 14.3.6, respectively.

Theorem 14.3.7. *Let $a \geq 0$. The sequences*

$$0 \longrightarrow \mathcal{O}^{(\text{exp}),a}(\mathbb{C}) \longrightarrow \mathcal{O}^{(\text{exp}),a} \xrightarrow{\text{bv}} \mathcal{U}'_{(t)}(\mathbb{C}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}^{\{\text{exp}\},a}(\mathbb{C}) \longrightarrow \mathcal{O}^{\{\text{exp}\},a}(\mathbb{C} \setminus \mathbb{R}) \xrightarrow{\text{bv}} \mathcal{A}'_{\{t\}}(\mathbb{R}) \longrightarrow 0$$

are exact (the boundary value operator being interpreted in the sense of Section 13.7.3).

Theorem 14.3.8. *Let $a \geq 0$.*

- (i) *Suppose that G is analytic in the half-plane $\text{Im } \zeta = \eta > \lambda$ for some $\lambda > 0$ and satisfies*

$$\sup_{\eta > \lambda} |G(\zeta)| e^{-(a+\varepsilon)|\eta| - (h+\varepsilon)|\xi|} < \infty \quad (14.3.8)$$

for some $h > 0$ and every $\varepsilon > 0$, then there is $f \in \mathcal{U}'_{(t),(-a)+}$ with $G = \mathcal{L}\{f; \cdot\}$. Conversely, let $f \in \mathcal{U}'_{(t),(-a)+}$. Then, $\mathcal{L}\{f; \cdot\}$ is analytic in some half-plane $\text{Im } \zeta = \eta > \lambda$ and satisfies (14.3.8) for some $h > 0$ and every $\varepsilon > 0$.

- (ii) *Suppose that G is analytic in the upper half-plane. Then, G satisfies (14.3.8) for every $h, \lambda, \varepsilon > 0$ if and only if there is $f \in \mathcal{A}'_{\{t\},(-a)+}$ with $G = \mathcal{L}\{f; \cdot\}$.*

Appendix A

Open problems

In the process of making this dissertation, we encountered several other interesting problems. Below we list the most important ones, and in some cases, ideas how to approach them.

Part I

- Let M_p be a weight sequence satisfying $(M.1)$, $(M.2)'$, (QA) , and (NA) . In analogy with Proposition 2.2.11, construct a flabby sheaf $\mathfrak{B}^{(M_p)}$ on \mathbb{R}^d such that, for all $K \Subset \mathbb{R}^d$, the space of global sections of $\mathfrak{B}^{(M_p)}$ with support in K is given by $\mathcal{E}'^{(M_p)}[K]$, that is, the space of quasianalytic functionals of class (M_p) supported in K . The main difference with the Roumieu case is that the space $\mathcal{E}'^{(M_p)}[K]$ is a (PLS) -space and not a Fréchet space. In fact, we have spent quite some time attempting to solve this problem but without success so far. Our first observation was that the proof of Lemma 2.2.13 is implicitly based on Lemma 2.3.6 (see [84, Thm. 1.2]). By following the same method as in [84, Thm. 1.2], but by making use of Theorem 2.3.7 instead of the Lemma 2.3.6, we were able to show the following analogue of Lemma 2.2.13 for *ultrabornological* projective limits of Montel (DF) -spaces.

Lemma. *Let X be a second countable, locally compact topological space. Assume that, for each relatively compact open subset U of X , a Montel (DF)-space $F(U)$ is given and that, for each inclusion $U_1 \subseteq U_2$ of relatively compact open subsets of X , there is an injective linear continuous mapping $\iota_{U_2, U_1} : F(U_1) \rightarrow F(U_2)$ such that $\iota_{U_3, U_1} = \iota_{U_3, U_2} \circ \iota_{U_2, U_1}$ and $\iota_{U_1, U_1} = \text{id}$ for all $U_1 \subseteq U_2 \subseteq U_3$. Define*

$$F_K := \varprojlim_{K \Subset U} F(U)$$

for all $K \Subset X$ and denote by ι_{K_2, K_1} the canonical inclusion mapping from F_{K_1} into F_{K_2} . Suppose that the following conditions are satisfied (we identify F_{K_1} with its image in F_{K_2} under the mapping ι_{K_2, K_1}):

- (FS0) *For all $K \Subset X$ the space F_K is ultrabornological.*
- (FS1) *Let $K \Subset X$ and let U be a relatively compact open neighbourhood of K such that every connected component of U meets K . Then, F_K is dense in $F(U)$.*
- (FS2) *For $K_1, K_2 \Subset X$ the mapping $F_{K_1} \times F_{K_2} \rightarrow F_{K_1 \cup K_2} : (f_1, f_2) \rightarrow f_2 - f_1$ is surjective.*
- (FS3) (i) *For $K_1, K_2 \Subset X$ it holds that $F_{K_1 \cap K_2} = F_{K_1} \cap F_{K_2}$.*
(ii) *Let $K_1 \supseteq K_2 \supseteq \dots$ be a decreasing sequence of compact sets in X and set $K = \bigcap_{n \in \mathbb{N}} K_n$. Then, $F_K = \bigcap_{n \in \mathbb{N}} F_{K_n}$.*
- (FS4) $F_\emptyset = \{0\}$.

Then, there exists an (up to sheaf isomorphism) unique flabby sheaf \mathcal{F} on X such that

$$\Gamma_K(X, \mathcal{F}) = F_K$$

for all $K \Subset X$.

By using similar arguments as in the proof of Proposition 2.2.11, the desired result would follow from the above lemma once we have shown that the (PLS) -space $\mathcal{E}^{(M_p)}[K]$ is ultra-bornological for each $K \Subset \mathbb{R}^d$. The latter is equivalent to the fact that the (LF) -space $\mathcal{E}^{(M_p)}[K]$ satisfies (wQ) (cf. Theorem 2.3.7). We tried several approaches to show this but all of them failed to now.

- Let M_p be a weight sequence satisfying $(M.1)$, $(M.2)'$, (QA) , and (NE) and let $\Omega \subseteq \mathbb{R}^d$ be open. Does the (FN) -space $\mathcal{E}^{(M_p)}(\Omega)$ have a Schauder basis? In this regard, we mention that Domański and Vogt showed that the (PLN) -space $\mathcal{A}(\Omega)$ does not admit a basis [58] while it is not known whether $\mathcal{E}^{\{M_p\}}(\Omega)$ has a Schauder basis (the latter question was explicitly raised by Domański and Vogt [17, p. 439]; see also [161]).
- One of the crucial results in Chapter 5 is that $\mathcal{A}(\Omega)$ has the dual interpolation estimate for small theta for *arbitrary* open subsets Ω of \mathbb{R}^d , as was shown by Bonet and Domański [17]. More generally, let M_p be a weight sequence satisfying $(M.1)$, $(M.2)'$, (QA) , and (NE) . Does $\mathcal{E}^{\{M_p\}}(\Omega)$ satisfy the dual interpolation estimate for small theta? For Ω convex and M_p satisfying $(M.4)$ ($= M_p/p!$ satisfies $(M.1)$) this was also proved in [17].
- Let M_p be a weight sequence satisfying $(M.1)$, $(M.2)$, $(M.2)^*$, (QA) , and (NE) and define the subsheaf $\mathcal{G}^{\{M_p\}, \infty}$ of the sheaf $\mathcal{G}^{\{M_p\}}$ in the same way as in the non-quasianalytic case (cf. Section 4.3). We believe that the equality

$$\mathcal{G}^{\{M_p\}, \infty}(\Omega) \cap \iota(\mathfrak{B}^{\{M_p\}}(\Omega)) = \mathcal{E}^{\{M_p\}}(\Omega)$$

should hold for all open subsets Ω of \mathbb{R}^d (where ι denotes the embedding from Theorem 6.4.4). We do not know how to

tackle this problem in general. However, the weaker statement

$$\mathcal{G}^{\{M_p\},\infty}(\Omega) \cap \iota(\mathcal{D}'(\Omega)) = \mathcal{E}^{\{M_p\}}(\Omega)$$

is already interesting. In this case, one could try to modify the proof of Theorem 4.3.2 by using a Paley-Wiener type result for ultradifferentiability of class $\{M_p\}$ due to Hörmander [79, Thm. 8.4.2] and by replacing single cut-off functions used in the proof by analytic cut-off sequences. We tried this approach but, somewhat to our surprise, some inherent technical difficulties appeared. Therefore, new ideas could possibly be needed.

- Let M_p be a weight sequence satisfying (M.1), (M.2), (QA), and (NE) (and possibly some extra conditions). Construct a sheaf of *diffeomorphism invariant* differential algebras containing $\mathfrak{B}^{\{M_p\}}$ as a linear differential subsheaf and $\mathcal{E}^{\{M_p\}}$ as a subsheaf of algebras. The fundamental difficulty to overcome is the fact that the space $\mathfrak{B}^{\{M_p\}}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open, cannot be realized as the dual of a test function space and that, consequently, there does not seem to be a direct way to regularize the elements of $\mathfrak{B}^{\{M_p\}}(\Omega)$. The latter principle lies at the heart of every diffeomorphism invariant Colombeau algebra constructed so far. In our opinion, this is one of the deepest and most fascinating problems that stems from our work.

Part II

- Extend the implication (iii) \Rightarrow (i) in Theorems 11.2.7 and 11.2.8 to weight sequences M_p that do not necessarily satisfy (M.3). Does (M.3)' suffice? Does this implication also hold in the quasianalytic case?

- Let M_p and A_p be two weight sequences satisfying (9.2). Let \mathcal{W} and \mathcal{V} be an increasing and decreasing weight system on \mathbb{R}^d , respectively, and assume that \mathcal{W} and \mathcal{V} are \dagger -admissible. Consider the following convolutor spaces

$$\mathcal{O}'_C(\mathcal{S}^*_\dagger, L^1_{\mathcal{W}}) := \{f \in \mathcal{S}'^*_\dagger(\mathbb{R}^d) \mid f * \varphi \in L^1_{\mathcal{W}}(\mathbb{R}^d) \\ \text{for all } \varphi \in \mathcal{S}^*_\dagger(\mathbb{R}^d)\}$$

and

$$\mathcal{O}'_C(\mathcal{S}^*_\dagger, L^1_{\mathcal{V}}) := \{f \in \mathcal{S}'^*_\dagger(\mathbb{R}^d) \mid f * \varphi \in L^1_{\mathcal{V}}(\mathbb{R}^d) \\ \text{for all } \varphi \in \mathcal{S}^*_\dagger(\mathbb{R}^d)\}.$$

Make an analysis of these spaces similar to the one that has been made for $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ in Chapter 10; this would considerably extend the results from Chapter 11. Our idea is to combine the methods developed in Chapters 10 and 11. For $\mathcal{O}'_C(\mathcal{S}^*_\dagger, L^1_{\mathcal{V}})$ the projective description of weighted inductive limits of L^1 -spaces from [139] will be indispensable.

Part III

- Develop an analytic representation theory for hyperfunctions and ultrahyperfunctions of fast growth in several variables. In particular, we are interested in analytic representations of (ultra)hyperfunctions of fast growth, the boundary value behaviour of analytic functions defined in general tube domains, and Epstein and Martineau type edge of the wedge theorems [142, 111].

Appendix B

Nederlandstalige samenvatting

In deze dissertatie worden drie onderwerpen in de theorie van ultradifferentieerbare functies en ultradistributies bestudeerd. Meer precies ontwikkelen we een niet-lineaire theorie voor ultradistributies en infrahyperfuncties, bestuderen de topologische eigenschappen van ruimten van convolutoren, en introduceren twee nieuwe klassen van gewogen ruimten van analytische functies en onderzoeken hun duale ruimten. Deze drie onderwerpen worden onafhankelijk van elkaar gepresenteerd.

In Deel I wordt een niet-lineaire theorie voor ultradistributies en infrahyperfuncties ontwikkeld. Het belangrijkste resultaat in dit deel is de constructie van een differentiaalalgebra die de ruimte van hyperfuncties bevat als een lineaire differentiële deelruimte en waarin het puntsgewijze product van reëel analytische functies behouden wordt. Dit resultaat lost een belangrijk probleem op in de niet-lineaire theorie van veralgemeende functies dat naar voren werd geschoven door Oberguggenberger [123, p. 286, Probleem 27.2]. Dit deel is gebaseerd op de artikels [40] (gezamenlijk werk met J. Vindas), [38, 39] (gezamenlijk werk met H. Vernaëve en J. Vindas), en [37] (gezamenlijk werk met E. A. Nigsch).

We bestuderen de lokaal convexe structuur van ruimten van convolutoren van Gelfand-Shilov ruimten in Deel II. In het bijzonder karakteriseren we wanneer deze ruimten bornologisch zijn. Ons werk beantwoordt de vraag gesteld in [50, p. 403]. Dit deel is gebaseerd op de artikels [42, 43, 44] (gezamenlijk werk met J. Vindas).

Ten slotte, introduceren we in Deel III twee nieuwe klassen van ruimten van analytische functies die naar nul naderen op horizontale stroken van het complexe vlak met respect tot een gewichtsfunctie. Hun duale ruimten veralgemenen de ruimten van Fourier hyperfuncties en Fourier ultrahyperfuncties. We ontwikkelen een analytische representatietheorie voor hun duale ruimten en passen deze dan toe om de niet-trivialiteit van deze ruimten van analytische functies te karakteriseren in termen van de definiërende gewichtsfunctie. Dit lost gedeeltelijk een beroemd open probleem op betreffende de niet-trivialiteit van algemene Gelfand-Shilov ruimten [63, Hfdst. 1]. Dit deel is gebaseerd op het artikel [41] (gezamenlijk werk met J. Vindas).

Bibliography

- [1] S. Abdullah, *On the space of convolution operators and multipliers in \mathcal{K}'_M* , J. Univ. Kuwait (Sci) **15** (1988), 219–227.
- [2] S. Abdullah, *On distributions of rapid growth*, Portugal. Math. **47** (1990), 191–204.
- [3] S. Abdullah, *On the topology of the space of convolution operators in \mathcal{K}'_M* , Proc. Amer. Math. Soc. **110** (1990), 177–185.
- [4] A. A. Albanese, *On completeness and projective descriptions of weighted inductive limits of spaces of Fréchet-valued continuous functions*, Math. Nachr. **216** (2000), 5–24.
- [5] C. Bargetz, *A sequence space representation of L . Schwartz' space \mathcal{O}_C* , Arch. Math. **98** (2012), 317–326.
- [6] C. Bargetz, N. Ortner, *Characterization of L . Schwartz' convolutor and multiplier spaces \mathcal{O}'_C and \mathcal{O}_M by the short-time Fourier transform*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **108** (2014), 833–847.
- [7] F. Bastin, *On bornological $C\overline{V}(X)$ spaces*, Arch. Math. **53** (1989), 394–398.
- [8] K. Benmeriem, C. Bouzar, *Generalized Gevrey ultradistributions*, New York J. Math. **15** (2009) 37–72.
- [9] K. Benmeriem, C. Bouzar, *An algebra of generalized Roumieu ultradistributions*, Rend. Sem. Mat. Univ. Politec. Torino **70** (2012), 101–109.

- [10] K. D. Bierstedt, *An introduction to locally convex inductive limits*, in: *Functional analysis and its applications*, pp. 35–133, World Scientific, Singapore, 1988.
- [11] K. D. Bierstedt, J. Bonet, *Weighted (LF) -spaces of continuous functions*, Math. Nachr. **165** (1994), 25–48.
- [12] K. D. Bierstedt, R. Meise, *Induktive Limites gewichteter Räume stetiger und holomorpher Funktionen*, J. Reine Angew. Math. **282** (1976), 186–220.
- [13] K. D. Bierstedt, R. Meise, W. H. Summers, *A projective description of weighted inductive limits*, Trans. Amer. Math. Soc. **272** (1982), 107–160.
- [14] G. Björck, *Linear partial differential operators and generalized distributions*, Ark. Mat. **6** (1966), 351–407.
- [15] J. Bonet, P. Domański, *Real analytic curves in Fréchet spaces and their duals*, Monatsh. Math. **126** (1998), 13–36.
- [16] J. Bonet, P. Domański, D. Vogt, *Interpolation of vector valued real analytic functions*, J. London Math. Soc. **66** (2002), 407–420.
- [17] J. Bonet, P. Domański, *The structure of spaces of quasianalytic functions of Roumieu type*, Arch. Math. (Basel) **89** (2007), 430–441.
- [18] J. Bonet, R. Meise, S. N. Melikhov, *A comparison of two different ways to define classes of ultradifferentiable functions*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 425–444.
- [19] J. Bonet, P. Pérez Carreras, *Barrelled locally convex spaces*, North-Holland Publishing Co., Amsterdam, 1987.
- [20] J.-M. Bony, P. Schapira, *Solutions hyperfonctions du problème de Cauchy*, in: *Hyperfunctions and pseudo-differential equations*, pp. 82–98, Springer-Verlag, New York, 1973.

- [21] R. W. Braun, R. Meise, B. A. Taylor, *Ultradifferentiable functions and Fourier analysis*, Result. Math. **17** (1990), 206–237.
- [22] G. Bredon, *Sheaf theory*, Second edition, Springer-Verlag, New York, 1997.
- [23] T. Carleman, *L'intégrale de Fourier et questions qui s'y rattachent*, Publications Scientifiques de l'Institut Mittag-Leffler, 1967.
- [24] R. D. Carmichael, A. Kamiński, S. Pilipović, *Boundary values and convolution in ultradistribution spaces*, World Scientific, Singapore, 2007.
- [25] R. Carmichael, D. Mitrović, *Distributions and analytic functions*, Longman Scientific & Technical, New York, 1989.
- [26] W. Chen, Z. Ditzian, *Mixed and directional derivatives*, Proc. Amer. Math. Soc. **108** (1990), 177–185.
- [27] S.-Y. Chung, D. Kim, S. Lee, *Characterization for Beurling-Björck space and Schwartz space*, Proc. Amer. Math. Soc. **125** (1997), 3229–3234.
- [28] J.-F. Colombeau, *New generalized functions and multiplication of distributions*, North-Holland Publishing Co., Amsterdam, 1984.
- [29] J.-F. Colombeau, *Elementary introduction to new generalized functions*, North-Holland Publishing Co., Amsterdam, 1985.
- [30] J.-F. Colombeau, *Nouvelles solutions d'équations aux dérivées partielles*, C. R. Acad. Sci. **301** (1985), 281–283.
- [31] F. Colombini, D. Del Santo, M. Reissig, *On the optimal regularity of coefficients in hyperbolic Cauchy problems*, Bull. Sci. Math. **127** (2003), 328–347.
- [32] F. Colombini, E. Jannelli, S. Spagnolo, *Well-posedness in the Gevrey classes of the Cauchy problem for a nonstrictly hyper-*

- holic equation with coefficients depending on time*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **10** (1983), 291–312
- [33] F. Colombini, S. Spagnolo, *An example of a weakly hyperbolic Cauchy problem not well posed in C^∞* , Acta Math. **148** (1982), 243–253.
- [34] J. B. Conway *Functions of one complex variable*, Springer-Verlag, New York, 1973.
- [35] N. Dapić, S. Pilipović, D. Scarpalézos, *Microlocal analysis of Colombeau’s generalized functions: propagation of singularities*, J. Anal. Math. **75** (1998), 51–66.
- [36] A. Debrouwere, *Generalized function algebras containing spaces of periodic ultradistributions*, in: *Generalized Functions and Fourier Analysis*, pp. 59–78, Birkhäuser, 2017.
- [37] A. Debrouwere, E. A. Nigsch, *Sheaves of nonlinear generalized function spaces*, New York J. Math. **23** (2017), 1751–1789.
- [38] A. Debrouwere, H. Vernaëve, J. Vindas, *Optimal embeddings of ultradistributions into differential algebras*, Monatsh. Math. (2017), in press, doi: 10.1007/s00605-017-1066-6.
- [39] A. Debrouwere, H. Vernaëve, J. Vindas, *A non-linear theory of infrahyperfuctions*, Kyoto J. Math., accepted, arXiv:1701.06996.
- [40] A. Debrouwere, J. Vindas, *Solution to the first Cousin problem for vector-valued quasianalytic functions*, Ann. Mat. Pura Appl. **196** (2017), 1983–2003.
- [41] A. Debrouwere, J. Vindas, *On the non-triviality of certain spaces of analytic functions. Hyperfunctions and ultra-hyperfunctions of fast growth*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A. Math. RACSAM (2017), in press, doi: 10.1007/s13398-017-0392-9.

- [42] A. Debrouwere, J. Vindas, *A projective description of generalized Gelfand-Shilov spaces of Roumieu type*, Proceedings of the 2017 ISAAC congress, accepted, arXiv:1710.11598.
- [43] A. Debrouwere, J. Vindas, *On weighted inductive limits of spaces of ultradifferentiable functions and their duals*, submitted, arXiv:1710.03731.
- [44] A. Debrouwere, J. Vindas, *Topological properties of convolution spaces via the short-time Fourier transform*, submitted, arXiv:1801.09246.
- [45] A. Delcroix, *Remarks on the embedding of spaces of distributions into spaces of Colombeau generalized functions*, Novi Sad J. Math. **35** (2005), 27–40.
- [46] A. Delcroix, M. F. Hasler, S. Pilipović, V. Valmorin, *Embeddings of ultradistributions and periodic hyperfunctions in Colombeau type algebras through sequence spaces*, Math. Proc. Cambridge Philos. Soc. **137** (2004), 697–708.
- [47] A. Delcroix, M. F. Hasler, S. Pilipović, V. Valmorin, *Sequence spaces with exponent weights. Realizations of Colombeau type algebras*, Dissertationes Math. **447** (2007), 56 pp.
- [48] J.-W. de Roeper, *Hyperfunctional singular support of ultradistributions*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1985), 585–631.
- [49] M. De Wilde, *Closed graph theorem and webbed spaces*, Pitman, London, 1978.
- [50] P. Dimovski, S. Pilipović, B. Prangoski, J. Vindas, *Convolution of ultradistributions and ultradistribution spaces associated to translation-invariant Banach spaces*, Kyoto J. Math. **56** (2016), 401–440.

- [51] P. Dimovski, S. Pilipović, J. Vindas, *New distribution spaces associated to translation-invariant Banach spaces*, Monatsh. Math. **177** (2015), 495–515.
- [52] P. Dimovski, B. Prangoski, D. Velinov, *Multipliers and convolutors in the space of tempered ultradistributions*, Novi Sad J. Math. **44** (2014), 1–18.
- [53] P. Dimovski, B. Prangoski, J. Vindas, *On a class of translation-invariant spaces of quasianalytic ultradistributions*, Novi Sad J. Math. **45** (2015), 143–175.
- [54] P. Domański, *Classical PLS-spaces: spaces of distributions, real analytic functions and their relatives*, in: Orlicz Centenary Volume, pp. 51–70. Banach Center Publications, Warszawa, 2004.
- [55] P. Domański, *Real analytic parameter dependence of solutions of differential equations*, Rev. Mat. Iberoam. **26** (2010), 175–238.
- [56] P. Domański, M. Langenbruch, *Vector valued hyperfunctions and boundary values of vector valued harmonic and holomorphic functions*, Publ. RIMS Kyoto Univ. **44** (2008), 1097–1142.
- [57] P. Domański, D. Vogt, *A splitting theory for the space of distributions*, Studia Math. **140** (2000), 57–77.
- [58] P. Domański, D. Vogt, *The space of real analytic functions has no basis*, Studia Math. **142** (2000), 187–200.
- [59] R. Estrada, J. Vindas, *On Borel summability and analytic functionals*, Rocky Mountain J. Math. **43** (2013), 895–903.
- [60] D. H. T. Franco, L. H. Renoldi, *A note on Fourier-Laplace transform and analytic wave front set in theory of tempered ultrahyperfunctions*, J. Math. Anal. Appl. **325** (2007), 819–829.

- [61] C. Garetto, *Topological structures in Colombeau algebras: topological \tilde{C} -modules and duality theory*, Acta Appl. Math. **88** (2005), 81–123.
- [62] C. Garetto, M. Ruzhansky, *Weakly hyperbolic equations with non-analytic coefficients and lower order terms*, Math. Ann. **357** (2013), 401–440.
- [63] I. M. Gelfand, G. E. Shilov, *Generalized functions. Vol. 3. Theory of differential equations*, Academic Press, New York, 1967.
- [64] I. M. Gelfand, G. E. Shilov, *Generalized functions. Vol. 2: Spaces of Fundamental and Generalized Functions*, Academic Press, New York, 1968.
- [65] R. Godement, *Topologie algébrique et théorie des faisceaux*, Third edition, Hermann, Paris, 1973.
- [66] A. Gorny, *Contribution à l'étude des fonctions dérivables d'une variable réelle*, Acta Math. **71** (1939), 317–358.
- [67] T. Gramchev, *Nonlinear maps in spaces of distributions*, Math. Z. **209** (1992), 101–114.
- [68] K. Gröchenig, *Foundations of time-frequency analysis*, Birkhäuser, Boston, 2001.
- [69] M. Grosser, E. Farkas, M. Kunzinger, R. Steinbauer, *On the foundations of nonlinear generalized functions I and II*, Mem. Amer. Math. Soc. **729**, 2001.
- [70] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer, *Geometric theory of generalized functions*, Kluwer Academic Publishers, Dordrecht, 2001.
- [71] M. Grosser, M. Kunzinger, R. Steinbauer, J. A. Vickers, *A Global Theory of Algebras of Generalized Functions*, Adv. Math. **166** (2002), 50–72.
- [72] M. Grosser, E. A. Nigsch, *Full and special Colombeau Algebras*, Proc. Edinb. Math. Soc., accepted, arXiv:1611.06061.

- [73] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16**, 1955.
- [74] R. C. Gunning, H. Rossi, *Analytic functions of several complex variables*, American Mathematical Society, Providence, 2009.
- [75] M. Hasumi, *Note on the n -dimensional tempered ultra-distributions*, Tôhoku Math. J. **13** (1961), 94–104.
- [76] T. Heinrich, R. Meise, *A support theorem for quasianalytic functionals*, Math. Nachr. **280** (2007), 364–387.
- [77] I. I. Hirschman, *On the behaviour of Fourier transforms at infinity and on quasi-analytic classes of functions*, Amer. J. Math. **72** (1950), 200–213.
- [78] L. Hörmander, *Between distributions and hyperfunctions*, Astérisque **131** (1985), 89–106.
- [79] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, Second edition, Springer-Verlag, Berlin, 1990.
- [80] G. Hörmann, *Hölder-Zygmund regularity in algebras of generalized functions*, Z. Anal. Anwendungen **23** (2004), 139–165.
- [81] G. Hörmann, M. Oberguggenberger, S. Pilipović, *Microlocal hypoellipticity of linear partial differential operators with generalized functions as coefficients*, Trans. Amer. Math. Soc. **358** (2006), 3363–3383.
- [82] J. Horváth, *Topological vector spaces and distributions*, Addison-Wesley, Reading, 1966.
- [83] R. F. Hoskins, J. Sousa Pinto, *Theories of generalised functions. Distributions, ultradistributions and other generalised functions*, Horwood Publishing Limited, Chichester, 2005.
- [84] Y. Ito, *Fourier hyperfunctions of general type*, J. Math. Kyoto Univ. **28** (1988), 213–265.

- [85] K. Junker, *Vektorwertige Fourierhyperfunktionen*, Diplomarbeit, Düsseldorf, 1978.
- [86] A. Kaneko, *Introduction to hyperfunctions*, Kluwer Academic Publisher, Dordrecht, 1988.
- [87] T. Kawai, *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Tokyo Sect. IA Math. **17** (1970), 467–517.
- [88] H. Komatsu, *On the index of ordinary differential operators*, J. Fac. Sci. Univ. Tokyo Sect IA Math. **18** (1971), 379–398.
- [89] H. Komatsu, *Ultradistributions I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20** (1973), 25–105.
- [90] H. Komatsu, *Relative cohomology of sheaves of solutions of differential equations*, in: Hyperfunctions and pseudo-differential equations, pp. 192–261, Springer, Berlin, 1973.
- [91] H. Komatsu. *Ultradistributions. II: The kernel theorem and ultradistributions with support in a submanifold*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1977), 607–628.
- [92] H. Komatsu, *Ultradistributions. III. Vector-valued ultradistributions and the theory of kernels*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), 653–717.
- [93] H. Komatsu, *Microlocal analysis in Gevrey classes and in complex domains*, in: Microlocal analysis and applications, pp. 161–236, Springer, Berlin, 1991.
- [94] P. Koosis, *Introduction to H_p spaces*, Second edition, Cambridge University Press, Cambridge, 1980.
- [95] J. Korevaar, *Tauberian theory. A century of developments*, Springer-Verlag, Berlin, 2004.
- [96] S. Kostadinova, S. Pilipović, K. Saneva, J. Vindas, *The short-time Fourier transform of distributions of exponential type and*

- Tauberian theorems for S -asymptotics*, FILOMAT **30** (2016), 3047–3061.
- [97] G. Köthe, *Topological vector spaces. I*, Springer-Verlag, New York, 1969.
 - [98] G. Köthe, *Die Randverteilungen analytischer Funktionen*, Math. Z. **57** (1952), 13–33.
 - [99] G. Köthe, *Dualität in der Funktionentheorie*, J. Reine Angew. Math. **191** (1953), 30–49.
 - [100] A. Kriegl, P. Michor, *the convenient setting for real analytic mappings*, Acta Math. **165** (1990), 105–159.
 - [101] A. Kriegl, P. W. Michor, *The convenient setting of global analysis*, American Mathematical Society, Providence, 1997.
 - [102] A. Kriegl, P. W. Michor, A. Rainer, *The convenient setting for quasianalytic Denjoy-Carleman differentiable mappings*, J. Funct. Anal. **261** (2011), 1799–1834.
 - [103] M. Langenbruch, *Ultradifferentiable functions on compact intervals*, Math. Nachr. **140** (1989), 109–126.
 - [104] M. Langenbruch, *Bases in spaces of analytic germs*, Ann. Polon. Math. **106** (2012), 223–242.
 - [105] J. Larcher, *Some remarks concerning the spaces of multipliers and convolutors, \mathcal{O}_M and \mathcal{O}_C , of Laurent Schwartz*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **106** (2012), 407–417.
 - [106] J. Larcher, J. Wengenroth, *A new proof for the bornologicity of the space of slowly increasing functions*, Bull. Belg. Math. Soc. Simon Stevin **5** (2014), 887–894.
 - [107] H. Lewy, *An example of a smooth partial differential equation without solution*, Ann. Math. **66** (1957), 155–158.
 - [108] S. Mandelbrojt, *Séries adhérents, régularisations des suites, applications*, Gauthier-Villars, Paris, 1952.

- [109] A. Martineau, *Les hyperfonctions de M. Sato*, Séminaire Bourbaki **6** (1960), 127–139.
- [110] A. Martineau, *Sur la topologie des espaces de fonctions holomorphes*, Math. Ann. **163** (1966), 62–88.
- [111] A. Martineau, *Théorèmes sur le prolongement analytique du type "edge of the wedge theorem"*, Séminaire Bourbaki **10** (1968), 445–461.
- [112] T. Matsuzawa, *A calculus approach to hyperfunctions I*, Nagoya Math. J. **108** (1987), 53–66.
- [113] R. Meise, B. A. Taylor, *Linear extension operators for ultradifferentiable functions of Beurling type on compact sets*, Amer. J. Math. **111** (1989), 309–337.
- [114] R. Meise, D. Vogt, *Introduction to functional analysis*, Clarendon Press, Oxford, 1997.
- [115] M. Morimoto, *Sur les ultradistributions cohomologiques*, Ann. Inst. Fourier (Grenoble) **19** (1969), 129–153.
- [116] M. Morimoto, *Analytic functionals with non-compact carrier*, Tokyo J. Math. **1** (1978), 77–103.
- [117] M. Morimoto, *An introduction to Sato's hyperfunctions*, American Mathematical Society, Providence, 1993.
- [118] M. Nedeljkov, S. Pilipović, D. Scarpalézos, *The linear theory of Colombeau generalized functions*, Longman Scientific & Technical, Harlow, 1998.
- [119] E. A. Nigsch, *Some extensions to the functional analytic approach to Colombeau algebras*, Novi Sad J. Math. **45** (2015), 231–240.
- [120] E. A. Nigsch, *The functional analytic foundation of Colombeau algebras*, J. Math. Anal. Appl. **421** (2015), 415–435.

- [121] E. A. Nigsch, *Nonlinear generalized sections of vector bundles*, J. Math. Anal. Appl. **440** (2016), 183–219.
- [122] E. A. Nigsch, *Colombeau Algebras without asymptotics*, J. Pseudo-Differ. Oper. Appl. (2017), in press, doi:10.1007/s11868-017-0230-z.
- [123] M. Oberguggenberger, *Multiplication of distributions and applications to partial differential equations*, Longman Scientific & Technical, Harlow, 1992.
- [124] Y. Oka, K. Yoshino, *Solvability of Lewy equation in the space of the tempered ultrahyperfunctions*, J. Pseudo-Differ. Oper. Appl. **3** (2012), 321–328.
- [125] N. Ortner, *On the spaces \mathcal{O}_C^m of John Horváth*, J. Math. Anal. Appl. **415** (2014), 62–74.
- [126] V. P. Palamodov, *The projective limit functor in the category of linear topological spaces*, Math. USSR-Sb. **4** (1968), 529–559.
- [127] V. P. Palamodov, *Homological methods in the theory of locally convex spaces*, Russian Math Surveys **26** (1971), 1–64.
- [128] Y. S. Park, M. Morimoto, *Fourier ultra-hyperfunctions in the Euclidean n -space*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20** (1973), 121–127.
- [129] H.-J. Petzsche, *Generalized functions and the boundary values of holomorphic functions*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 391–431.
- [130] H.-J. Petzsche, *On E. Borel’s theorem*, Math. Ann. **282** (1988), 299–313.
- [131] H.-J. Petzsche, D. Vogt, *Almost analytic extension of ultradifferentiable functions and the boundary values of holomorphic functions*, Math. Ann. **267** (1984), 17–35.

- [132] S. Pilipović, *Characterization of bounded sets in spaces of ultradistributions*, Proc. Amer. Math. Soc. **120** (1994), 1191–1206.
- [133] S. Pilipović, *Microlocal analysis of ultradistributions*, Proc. Amer. Math. Soc. **126** (1998), 105–113.
- [134] S. Pilipović, B. Prangoski, J. Vindas, *On quasianalytic classes of Gelfand-Shilov type. Parametrix and convolution*, J. Math. Pures Appl., in press, doi:10.1016/j.matpur.2017.10.008.
- [135] S. Pilipović, D. Scarpalézos, *Colombeau generalized ultradistributions*, Math. Proc. Camb. Phil. Soc. **130** (2001), 541–553.
- [136] S. Pilipović, D. Scarpalezos, *Regularity properties of distributions and ultradistributions*, Proc. Amer. Math. Soc. **129** (2001), 3531–3537.
- [137] K. Piszczek, *On a property of PLS-spaces inherited by their tensor products*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 155–170.
- [138] B. Prangoski, *Laplace transform in spaces of ultradistributions*, Filomat **27** (2013), 747–760.
- [139] K. Reiher, *Weighted inductive and projective limits of normed Köthe function spaces*, Results Math. **13** (1988), 147–161.
- [140] T. Rösner, *Surjektivität partieller Differentialoperatoren auf quasianalytischen Roumieu-Klassen*, Diplomarbeit, Düsseldorf, 1997.
- [141] C. Roumieu, *Sur quelques extensions de la notion de distribution*, Ann. Sci. École Norm. Sup. Sér 3 **77** (1960), 41–121.
- [142] W. Rudin, *Lectures on the edge-of-the-wedge theorem*, , Providence, 1971.
- [143] W. Rudin, *Real and complex analysis*, Third edition, McGraw-Hill, Singapore, 1987.

- [144] G. Sampson, Z. Zielezny, *Hypoelliptic convolution equations in \mathcal{K}'_p , $p > 1$* , Trans. Amer. Math. Soc. **223** (1976), 133–154.
- [145] M. Sato, *Theory of hyperfunctions I, II*, J. Sci. Univ. Tokyo Sect. I **8** (1959), 139–193; **8** (1960), 387–437.
- [146] P. Schapira, *Une équation aux dérivées partielles sans solutions dans l'espace des hyperfonctions*, C. R. Acad. Sci. Paris Sér. A **265** (1967), 665–667.
- [147] P. Schapira, *Théorie des hyperfonctions*, Springer-Verlag, 1970.
- [148] L. Schwartz, *Espaces de fonctions différentiables à valeurs vectorielles*, J.d'Analyse Math. **4** (1954), 88–148.
- [149] L. Schwartz, *Sur l'impossibilité de la multiplication des distributions*, C. R. Acad. Sci. Paris **239** (1954), 847–848.
- [150] L. Schwartz, *Théorie des distributions à valeurs vectorielles. I*, Ann. Inst. Fourier (Grenoble) **7** (1957), 1–141.
- [151] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.
- [152] J. Sebastião e Silva, *Les fonctions analytiques comme ultradistributions dans le calcul opérationnel*, Math. Ann. **136** (1958), 58–96.
- [153] J. Sousa Pinto, *Silva tempered ultradistributions*, Portugal. Math. **47** (1990), 267–292.
- [154] T. Takiguchi, *Structure of quasi-analytic ultradistributions*, Publ. Res. Inst. Math. Sci. **43** (2007), 425–442.
- [155] T. Takiguchi, *On the structure of hyperfunctions and ultradistributions*, Publ. Res. Inst. Math. Sci. Kokyuroku **1861** (2013), 71–82.
- [156] F. Trèves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.

- [157] V. Valmorin, *Generalized hyperfunctions on the circle*, J. Math. Anal. Appl. **261** (2001), 1–16.
- [158] D. Vogt, *On the functors $\text{Ext}^1(E, F)$ for Fréchet spaces*, Studia Math. **85** (1987), 163–197.
- [159] D. Vogt, *Topics on projective spectra of (LB) -Spaces*, in: Advances in the Theory of Fréchet Spaces, pp. 11–27, Springer, Dordrecht, 1989.
- [160] D. Vogt, *Regularity properties of (LF) -spaces*, in: Progress in functional analysis, pp. 57–84, North-Holland, Amsterdam, 1992.
- [161] D. Vogt, *Linear topological properties of spaces H^ω and of spaces of ultradifferentiable functions*, in: Functional Analysis and Complex Analysis, pp. 145–158, AMS, 2009.
- [162] D. Vogt, M. J. Wagner, *Charakterisierung der Unterräume und Quotientenräume der nuklearen stabilen Potenzreihenräume von unendlichem Typ*, Studia Math. **70** (1981), 63–80.
- [163] J. Wengenroth, *Acyclic inductive spectra of Fréchet spaces*, Studia Math. **120** (1996), 247–258.
- [164] J. Wengenroth, *Derived functors in functional analysis*, Springer-Verlag, Berlin, 2003.
- [165] D. V. Widder, *Functions harmonic in a strip*, Proc. Amer. Math. Soc. **12** (1961), 67–72.
- [166] Z. Zielezny, *On the space of convolution operators in \mathcal{K}'_1* , Studia Math. **31** (1968), 111–124.
- [167] V. V. Zharinov, *Fourier-ultrahyperfunctions*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), 533–570; English translation: Math. USSR-Izv. **16** (1981), 479–511.